



# DIPLOMARBEIT

Titel der Diplomarbeit

**On a general 3+1 formalism: Electromagnetism and  
ideal hydrodynamics in curved spacetime**

angestrebter akademischer Grad

**Magistra der Naturwissenschaften (Mag. rer. nat.)**

Verfasser:	Patricia Schmidt, Bakk.rer.nat.
Matrikel-Nummer:	0400863
Studienrichtung (lt. Studienblatt):	A 411 Diplomstudium Physik UniStG
Betreuer:	Ao. Univ.-Prof. Dr. Robert Beig

Wien, am 5. Mai 2010



# Abstract

This thesis studies and derives a general  $3 + 1$  formalism in the context of the initial value formulation of electromagnetism and ideal hydrodynamics in curved spacetime: From a strictly mathematical point of view it is interesting to consider a general slicing of spacetime instead of a regular foliation [13]. Whilst a foliation of spacetime requires that its leaves, labelled  $\varphi_t(\Sigma)$ , do not intersect and that their disjoint union covers the entire manifold  $M$ , in a general slicing the intersection of individual hypersurfaces is allowed and  $M$  does not necessarily have to be covered completely. This implies that the lapse function  $N$  can take any value and does not have to be constrained to strictly positive values. One of the crucial consequences is that a natural tensor field like the unit normal vector field  $\mathbf{n}$  is no longer defined on the spacetime manifold anymore, but only along the family of embeddings. In more fancy words, the unit normal is not anymore a section of the tangent bundle  $\mathcal{T}M$  over  $M$ , but only a section of the pullback bundle of  $\mathcal{T}M$  under the map  $\mathbb{R} \times \Sigma \ni (t, y) \times \Sigma \mapsto p = \varphi_t(y) \in M$ . Consequently one has to be far more careful when deriving the time evolution equations of geometrical and/or physical objects. Nonetheless, we develop a general concept such, that meaningful interpretations of conceptually difficult expressions can be given.

After a detailed explanation of the differential geometric framework, we derive the  $3 + 1$  formulation of Maxwell's theory of electromagnetism and of an ideal fluid in full detail. These two models are the two main ingredients of the most relevant model in Astrophysics, namely of magnetohydrodynamics. Therefore they are essential for the description of astrophysical objects like stars, accretion disks etc. [12]. Concerning the model of a perfect fluid, we derive the evolution equations for the standard Eulerian and the Taub formulations.

Afterwards, we apply the  $3 + 1$  splitting to Einstein's field equations of General Relativity and therefore obtain the well-established evolution and constraint equations, the ADM-equations.

We present the derivation of the  $3 + 1$  evolution equations of special mat-

ter models in a completely general setting and do not impose any restrictions on lapse and shift. Furthermore, the usage of Riemannian neighbourhoods in the spacetime manifold  $M$  can be avoided completely and therefore the derivation is done in full generality. Although the existence of solutions in the case of a vanishing lapse has not been proven yet, these more general time evolution schemes for certain dynamical systems in curved spacetime are interesting for specific slicings, e.g., CMC-slicings in cosmology [4, 5]

# Zusammenfassung

Diese Diplomarbeit leitet einen allgemeinen  $3 + 1$  Formalismus im Zusammenhang mit der Anfangswertproblem-Formulierung des Elektromagnetismus und der idealen Hydrodynamik in einer gekrümmten Raumzeit her und studiert diesen im Detail: Von einem rein mathematischen Standpunkt aus ist es sehr interessant ein so genanntes Slicing an Stelle einer Foliation zu betrachten [13]. Während die einzelnen Blätter einer Foliation einander nicht schneiden dürfen und daher als disjunkte Vereinigung die gesamte Raumzeit-Mannigfaltigkeit überdecken, sind Schnitte bei der Betrachtung eines Slicings gestattet. Mathematisch drückt sich dies dadurch aus, dass die beiden kinematischen Größen lapse  $N$  und shift  $\beta$  beliebige Werte annehmen können. Insbesondere kann der lapse daher auch verschwinden. Eine bedeutende Konsequenz dieses allgemeinen Rahmens ist, dass natürliche Tensorfelder wie z.B. das Einheitsnormalenvektorfeld  $\mathbf{n}$  nicht mehr länger auf der Raumzeit selbst definiert sind, sondern nur noch entlang der gegebenen Familie von Einbettungen. Mathematisch bedeutet dies, dass  $\mathbf{n}$  kein Schnitt des Tangentialbündels  $\mathcal{T}M$  von  $M$  ist, sondern ein Schnitt des Pullback-Bündels von  $\mathcal{T}M$  unter der Abbildung  $\mathbb{R} \times \Sigma \ni (t, y) \times \Sigma \mapsto p = \varphi_t(y) \in M$ . Folglich ist bei der Herleitung der Zeitentwicklungsgleichungen physikalischer und/oder geometrischer Größen erhöhte Sorgfalt geboten. Wir entwickeln hier ein allgemeines Konzept, sodass konzeptuell schwierige Begriffe sinnvoll interpretiert werden können.

Nach einer detaillierten Erklärung des geometrischen Rahmens wird die Anfangswertproblem-Formulierung des Elektromagnetismus und der idealen Hydrodynamik hergeleitet. Diese beiden Modelle zusammen sind die beiden Hauptbestandteile des in der Astrophysik am häufigsten verwendeten Materiemodells: der Magnetohydrodynamik [12]. Wir betrachten das Modell des idealen Fluids in seinen zwei gängigsten Varianten, nämlich in der üblichen Euler-Formulierung und der sogenannten Taub-Formulierung.

Im Anschluss daran wenden wir den  $3 + 1$  Formalismus direkt auf die Einsteinschen Feldgleichungen der Allgemeinen Relativitätstheorie an und erhalten so die  $3 + 1$  Entwicklungsgleichungen, so wie die Zwangsbedingungen, die zusammen als ADM-Gleichungen bekannt sind.

Wir leiten die  $3 + 1$  Entwicklungsgleichungen spezieller Materiemodelle in einem vollkommen allgemeinen Rahmen her und unterwerfen lapse und shift keinen Zwangsbedingungen. Zusätzlich verzichten wir auf die Verwendung Riemannscher Normalumgebungen. Daher erfolgt die Herleitung in absoluter Allgemeinheit und ohne Einschränkungen. Obwohl die Existenz von Lösungen im Fall eines verschwindenden lapse nicht bekannt ist, ist diese allgemeine Formulierung bei der Betrachtung bestimmter Slicings von Interesse [4, 5].

# Acknowledgements

Let me use this page to express my gratitude to several people for their continuous support and encouragement throughout the last years. They always tried to take some time to listen to me, though they were very busy themselves and had to hold down their own work.

First, I want to thank Prof. Dr. Robert Beig, the supervisor of this thesis, for entrusting me with this topic and supporting me during the entire time of calculation and writing.

Many thanks go to Dr. Mark Heinzle, who always found time to discuss upcoming questions with me. Furthermore, Mark always cheered me up when I was stuck and led me back on the right track. There are hardly any adequate words to express my thankfulness and to emphasise his relevance for this work.

Then, I would like to show my gratitude to Michael Bartel, Mag. Nicolai Friis, Mag. Johanna Akbarzadeh and Mag. Teresa Torzicky, who always cheered me up and believed in my skills, when I thought I was not doing the right thing. Many thanks go to Mag. Stephan Broda and Mag. Günther Waxenegger, two colleagues and friends, for the many fruitful discussions and distractions. Special thanks go to Prof. Dr. Helmuth Urbantke for answering any question concerning differential geometry. Additionally, I want to thank Stephan and Prof. Urbantke for the good working atmosphere in our office. I also want to show courtesy to Dr. Stephan Puchegger and Dr. Mark Hannam, who showed me lots of  $\text{\LaTeX}$  tricks and helped me many times typesetting equations and implementing graphics. It would have taken me much longer to finish this thesis without their help. Special thanks go to Dr. Mark Hannam for many discussions on slicing etc. and for proofreading the thesis.

I am indebted to my best friend Miriam Kollmann, who is always there for me when I need her, cheering me up, making me laugh and taking me out for a good chat and drinks. Our friendship is one of the most valuable things I own.

And, finally, I would like to thank my parents and my stepfather Gerhard, who endured my changing moods, my strain and stress especially during the phase of writing. Their absolute support and belief in me is the basis of all my work and studies.





# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Notation and Conventions . . . . .	3
<b>2</b>	<b>Hypersurfaces</b>	<b>4</b>
2.1	Lorentzian Spacetime . . . . .	4
2.2	Submanifolds . . . . .	6
2.3	Geometry of Hypersurfaces . . . . .	8
2.3.1	Definition . . . . .	8
2.3.2	Unit normal . . . . .	9
2.3.3	Intrinsic curvature of $\varphi(\Sigma)$ . . . . .	10
2.3.4	Extrinsic curvature of $\varphi(\Sigma)$ . . . . .	11
2.3.5	Projection onto $\Sigma$ . . . . .	12
2.3.6	The Codazzi relation . . . . .	15
2.3.7	The Gauß relation . . . . .	16
2.3.8	Examples of Hypersurfaces . . . . .	18
2.4	Kinematics of Hypersurfaces . . . . .	20
2.4.1	Time evolution vector field . . . . .	21
2.4.2	Slicing vs. foliation . . . . .	24
2.4.3	Evolution of the spatial metric . . . . .	26
<b>3</b>	<b>The 3 + 1 Equations for the Electromagnetic Field</b>	<b>28</b>
3.1	Introduction . . . . .	28
3.2	Evolution of the Electric Field . . . . .	30
3.3	Evolution of the Magnetic Field . . . . .	33
3.4	Projected Divergence Equations . . . . .	37
3.5	The 3+1 Maxwell system . . . . .	38
<b>4</b>	<b>The 3+1 Formulation of General Relativistic Hydrodynamics</b>	<b>40</b>
4.1	Introduction . . . . .	40
4.2	Fluid kinematics and thermodynamics . . . . .	41
4.3	The 3+1 Euler equations . . . . .	43
4.4	The Taub formulation . . . . .	50

<b>5</b>	<b>The 3 + 1 Formulation of the Einstein Equations</b>	<b>56</b>
5.1	Projection of Einstein's equations . . . . .	57
5.2	The 3+1 Einstein system . . . . .	60
5.3	The Initial Value Problem in General Relativity . . . . .	60
<b>6</b>	<b>Summary</b>	<b>65</b>
	<b>Concluding Remarks</b>	<b>68</b>
<b>A</b>	<b>Lie Derivative</b>	<b>70</b>
<b>B</b>	<b>Time derivative of the unit normal <math>n</math></b>	<b>74</b>
<b>C</b>	<b>Evolution equation for <math>K</math> - an alternative derivation</b>	<b>78</b>
<b>D</b>	<b>The stress-energy tensor</b>	<b>82</b>
	<b>References</b>	<b>85</b>
	<b>Curriculum Vitae</b>	<b>89</b>

# Chapter 1

## Introduction

Specific, simplifying models are used to describe baryonic matter distributions in the entire universe. Considering well-known astrophysical scenarios like the formation of accretion disks around black holes, it is natural to ask about the dynamics of such a lump of matter in the neighbourhood of a given strong gravitational field. In short, we ask how the dynamic of matter evolves in a given curved background.

It is well-known from classical physics, that the dynamics and thus the evolution of a physical system is governed by differential equations. In the context of classical physics, e.g. mechanics, these equations are *ordinary differential equations* (ODE), whereas Einstein's equations as well as all other field equations are *partial differential equations* (PDE).

In order to illustrate how something abstract like a differential equation can completely describe the motion of a system, let us assume a simple mechanical problem: consider a stone being thrown upwards in the Earth's Newtonian gravitational field. Then, one knows already from an undergraduate course in mechanics, that its motion can be determined completely, if the initial position and the initial velocity of the stone are given. Hence, if the initial values are known and well-posed, the problem possesses a unique solution. Although “only” an ODE has to be considered, this example should provide an easy understandable analog to the far more complex case of matter in a given spacetime. In order to state something about the dynamical evolution of matter in curved spacetime, it thus is important to reformulate the dynamical equations as an *initial value problem*. The so-called  $3 + 1$  formalism is one method that allows one to pose field equations in such a way. For this purpose, the 4-dimensional spacetime manifold is split into 3-dimensional hypersurfaces representing “space”, and “time”. This decomposition into space and time admits one to manipulate time-varying quantities in the ordinary 3-dimensional space. The time evolution we are

interested in is then given by propagation of initial data, i.e. appropriate data given at the time  $t = 0$ , along the time direction.

The  $3 + 1$  formalism was originally developed by André Lichnerowicz in the 1940's [29, 30, 31] and his PhD student Yvonne Choquet-Bruhat in their studies of the initial value formulation of General Relativity itself. In 1952, Choquet-Bruhat proved the existence of a locally unique solution of the Cauchy problem arising from the  $3 + 1$  decomposition [17, 18]. In the 1950's and 60's Dirac, Wheeler, as well as Arnowitt, Deser, Misner (ADM) and others further developed the  $3 + 1$  splits as part of their efforts to establish a Hamiltonian formulation of General Relativity. It was this so-called *ADM-formulation* that lay the foundation for a *canonical quantisation* of the gravitational field.

Since the 1970's the full ADM  $3 + 1$  formalism has become the basic tool in Numerical Relativity. James W. York developed a general method to solve the initial value problem and put the governing equations in the shape commonly used by the Numerical Relativity community. Nowadays, most numerical codes for solving Einstein's equations are based on the  $3 + 1$  formalism.

The subject of this thesis is to deduce the  $3+1$  formulation of electromagnetism and ideal hydrodynamics in curved spacetime in the *mathematically most general way*. Thus, we allow arbitrary *lapse* and *shift* and do not impose any constraints on them. Therefore, we shall start with a detailed explanation of the geometrical framework, which we then apply to certain field theories in order to gain valid initial value formulations.

*Chapter 2* is dedicated to the geometry of a single hypersurface being embedded into the 4-dimensional spacetime manifold and its evolution due to the consideration of a family of embeddings generated by a local vector field. The relevant kinematic quantities like lapse  $N$  and shift  $\beta$  are introduced in this chapter. The two subsequent chapters then deal with the  $3 + 1$  decomposition of the electromagnetic field equations and the Euler equations governing the dynamics of a perfect fluid. We have chosen these two models since their combination yields the most astrophysically relevant model, namely ideal magnetohydrodynamics. Furthermore, we present the  $3+1$  splitting of Taub's formulation of general relativistic hydrodynamics. In *Chapter 5* the explicit decomposition of the Einstein field equations, where the Cauchy problem with constraints originates from, is done. We will see that the essential dynamical variables are the spatial metric  $\mathbf{h}$  and the extrinsic curvature  $\mathbf{K}$ . The last section of this chapter gives a brief overview of the nature of the initial value formulation of General Relativity, mentioning the crucial theorems and concepts to obtain well-posedness. *Chapter 6*

gives a brief summary of the results we have obtained and *Chapter 7* contains some final, concluding remarks.

In order to not disturb the logical structure of this work, useful tools, mathematical concepts and alternative, cumbersome derivations are not present throughout the chapters but can be found in the appendices at the end of the thesis. For this purpose, *Appendix A* summarises the essential concept of the Lie dragging and Lie derivative. *Appendix B* obtains the time derivative of the normal vector field  $\mathbf{n}$  in the context of a general slicing, which is of crucial relevance. *Appendix C* contains a conceptually elegant but elaborate derivation of the evolution equations of the extrinsic curvature. Since certain projections of the geometric quantity describing matter in a full general relativistic context, the stress-energy tensor, are required, *Appendix D* is dedicated to the general  $3 + 1$  decomposition of it.

## 1.1 Notation and Conventions

Unfortunately, there exists no standard notation or sign convention within the General Relativity community. Thus, it is worth briefly mentioning the conventions used here.

When talking about a Lorentzian manifold, we always refer to a 4-dimensional spacetime of signature  $sign(-, +, +, +)$ , unless otherwise denoted. Spacetime coordinates are labelled by greek letters which run from 0 to 3, while purely spatial coordinates will be assigned by latin indices and thus only run from 1 to 3.

Furthermore, we use Einstein's convention of summation throughout the whole thesis. Thus, wherever two equal indices, one covariant the other contravariant, appear a summation is performed.

We also use so-called *geometric units*, where the speed of light  $c$  as well as the gravitational constant  $G$  are set to unity. In the context of general relativistic hydrodynamics, we also consider the Boltzmann constant  $k_B$  to be of dimension one.

## Chapter 2

# Hypersurfaces

As the notion of *hypersurfaces* is crucial to understand the 3+1 formalism of General Relativity, we dedicate the entire chapter to them. We focus on the geometric and kinematic properties of single hypersurfaces, thus everything discussed here is completely independent of the Einstein field equations.

We follow the structure and representation of Éricourgoulhon's notes [20] although there are essential differences concerning the differential geometric frame and the notation. If the reader is interested in a further mathematical presentation, we refer to O'Neill's book on semi-Riemannian geometry [33].

### 2.1 Lorentzian Spacetime

As we are interested in physically relevant models of 4-dimensional spacetimes, we consider a *smooth* ( $\mathcal{C}^\infty$ ) **Lorentzian** manifold  $M$ . These types of manifolds  $(M, g)$  are characterised by a non-degenerate metric  $\mathbf{g}$  of signature  $sign = (-, +, +, +)$ . Furthermore, we assume that  $(M, g)$  is *time orientable* and *torsion free*. The unambiguous *affine connection* of our spacetime, the **Levi-Civita connection**, is denoted by  $\nabla$ . Note that this symbol refers to the **spacetime connection** only. Other connections introduced later are denoted by a different symbol in order to avoid any possible confusions.

Consider a given point  $p \in M$ . The **tangent space** of  $M$  at this point  $p$  is symbolised by  $\mathcal{T}_p(M)$ . It is the space of all vectors tangent to  $M$  at  $p$ . Its dual, the **cotangent space** of  $M$  at  $p$ , is denoted by  $\mathcal{T}_p^*(M)$  and consists of all linear forms at  $p$ . In general, a covariant tensor over a vector space can be viewed as a multilinear map of this vector space into the real numbers  $\mathbb{R}$  [36].

*Remark:* We do not use *abstract index notation* [42] but we use *index notation* in order to handle calculations explicitly. Therefore, let us recall how

the components  $T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$  of a type  $(p, q)$  tensor  $\mathbf{T}$  are given w.r.t. the bases  $(\mathbf{e}_\mu)$  and  $(\mathbf{e}^\nu)$ :

$$\mathbf{T} = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mathbf{e}_{\mu_1} \otimes \dots \otimes \mathbf{e}_{\mu_p} \otimes \mathbf{e}^{\nu_1} \otimes \dots \otimes \mathbf{e}^{\nu_q}. \quad (2.1)$$

One of the most important operations is the **covariant derivative**  $\nabla$  of a tensor field  $\mathbf{T}$ . Its components are defined by:

$$\nabla \mathbf{T} = \nabla_\sigma T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mathbf{e}_{\mu_1} \otimes \dots \otimes \mathbf{e}_{\mu_p} \otimes \mathbf{e}^{\nu_1} \otimes \dots \otimes \mathbf{e}^{\nu_q} \otimes \mathbf{e}^\sigma. \quad (2.2)$$

As a very useful result, the covariant derivative of a tensor  $\mathbf{T}$  along a vector field  $\mathbf{v}$  is related to  $\nabla T$  by:

$$\nabla_v \mathbf{T} = v^\sigma \nabla_\sigma T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mathbf{e}_{\mu_1} \otimes \dots \otimes \mathbf{e}_{\mu_p} \otimes \mathbf{e}^{\nu_1} \otimes \dots \otimes \mathbf{e}^{\nu_q}. \quad (2.3)$$

Its components are then  $v^\sigma \nabla_\sigma T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$ .

As already noted before, a spacetime manifold is mainly described by its **metric**  $\mathbf{g}$ , which is a symmetric bilinear form  $g : \mathcal{T}_p(M) \times \mathcal{T}_p(M) \mapsto \mathbb{R}$  such, that  $\forall \mathbf{v}, \mathbf{w} \in \mathcal{T}_p(M)$   $\mathbf{g}(\mathbf{u}, \mathbf{w}) = g_{\mu\nu} u^\mu w^\nu$ . This denotes the *scalar product* between two vectors and is often written as the dot product  $\mathbf{u} \cdot \mathbf{w}$ .

Aside from this, the metric  $\mathbf{g}$  induces an isomorphism between  $\mathcal{T}_p(M)$  and  $\mathcal{T}_p^*(M)$ , which corresponds to the raising and the lowering of indices by applying either  $g_{\mu\nu}$  or the *inverse metric*  $g^{\mu\nu}$ , defined such that

$$g_{\alpha\mu} g^{\alpha\nu} = \delta_\mu^\nu, \quad (2.4)$$

where  $\delta_\mu^\nu$  denotes the standard *Kronecker symbol*. Thus, vectors can be transformed into 1-forms and vice versa. Such an operation is called a **metric contraction**. *Contraction* itself is another important basic operation in differential geometry. Consider, for instance, a second rank covariant tensor  $A_{\mu\nu}$  and a vector  $v^\nu$ , then the contraction of  $\mathbf{A}$  with  $\mathbf{v}$  is given by  $A_{\mu\nu} v^\nu = a_\mu \in \mathcal{T}_p^*(M)$  in index notation.

One of the crucial facts in General Relativity compared to Special Relativity or Newtonian physics is, that the spacetime manifold one operates on, is not flat anymore but curved due to existing gravitational fields. Hence, the existence of gravity is encoded in the geometry of spacetime. Far away from gravitational field sources, spacetime becomes Minkowskian, i.e. flat as known from Special Relativity, but close to stars for example, spacetime is curved. The differential geometrical object to describe the spacetime **curvature** is the **Riemann curvature tensor**  ${}^4R$ , which is defined as follows:

**Definition 1** *Let  $M$  be a semi-Riemannian manifold with Levi-Civita connection  $\nabla$ . The map  ${}^4R : \mathcal{X}(M)^3 \rightarrow \mathcal{X}(M)$ ,  ${}^4R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$   $\forall X, Y, Z \in \mathcal{X}(M)$ , is a  $(1, 3)$ -tensor field, the so-called **Riemann curvature tensor**  ${}^4R$  of  $(M, \nabla)$ , where  $\mathcal{X}(M)$  denotes the set of smooth vector fields over  $M$  and  $[\cdot, \cdot]$  the standard Lie bracket of vector fields.*

As before, the components of this tensor field in a given basis ( $\mathbf{e}_\alpha$ ) are denoted by  ${}^4R^\mu{}_{\nu\gamma\sigma}$ . Since  ${}^4R$  is a tensor field, single tangential vectors can be used instead of vector fields:  $x, y \in \mathcal{T}_p(M)$   ${}^4R_{xy} : \mathcal{T}_p(M) \rightarrow \mathcal{T}_p(M)$  such, that  $z \mapsto {}^4R_{xy}z$ . The above definition of the curvature tensor then yields an important identity, the **Ricci identity**:

$$\forall v \in \mathcal{T}_p(M), \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\sigma = {}^4R^\sigma{}_{\rho\mu\nu}v^\rho. \quad (2.5)$$

Many textbooks refer to Eq.(2.5) as the defining equation for the curvature tensor. Notice from Definition (1), that the Riemann curvature tensor is symmetric on the exchange of the two pairs of indices, whereas it is anti-symmetric on the exchange of the indices within a pair. Additionally, it satisfies the cyclic property, often called the **first Bianchi identity**:

$${}^4R_{xy}z + {}^4R_{yz}x + {}^4R_{zx}y = 0. \quad (2.6)$$

Contraction of the Riemann tensor leads to another essential geometric object: the **Ricci tensor**.

**Definition 2** *Let  ${}^4R$  be the Riemann curvature tensor of a semi-Riemannian manifold  $M$ . The **Ricci tensor**  ${}^4Ric$  of  $M$  is the  $(1,3)$  contraction of  ${}^4R$ . In local coordinates this is denoted by:*

$${}^4R_{\mu\nu} = {}^4R^\sigma{}_{\mu\sigma\nu}. \quad (2.7)$$

*Remark:* We use here the common notation for the Riemann and the Ricci tensor. Both tensors are indicated by the letter  $R$ , but the number of indices allows to distinguish between the two tensors unambiguously.

Finally, we are now able to define the **scalar curvature** of a manifold, which is the characterising scalar quantity of gravity in the context of General Relativity:

**Definition 3** *The **scalar curvature**  ${}^4\mathcal{R}$ , sometimes also called the **Ricci curvature**, of a Lorentzian manifold  $M$  is the metric contraction of its Ricci tensor. In local coordinates this writes as*

$${}^4\mathcal{R} = g^{\mu\nu}{}^4R_{\mu\nu} = tr({}^4R_{\mu\nu}) = {}^4R^\mu{}_\mu. \quad (2.8)$$

## 2.2 Submanifolds

Since we have described the most important properties of Lorentzian manifolds in Sec. 2.1, we now focus on **submanifolds** in this passage.

**Definition 4** *A **submanifold**  $N$  of a manifold  $M$  is a subset which itself has the structure of a manifold and for which the inclusion map satisfies certain properties.*



In the following we consider a Lorentzian spacetime  $(M, g, \nabla)$ , a submanifold  $N$  and an inclusion map  $\varphi : N \hookrightarrow M$  such, that  $N \subseteq M$ . If  $N$  additionally is a topological subspace in  $M$ , i.e. if the natural topology of  $N$  is the trace topology of  $M$ ,  $N$  is called a *regular submanifold*. The map  $\varphi$  induces a scalar product  $h = \varphi^*g$  on  $N$ , where  $\varphi^*$  denotes the *pullback* (see Definition 8 below) of the spacetime metric  $g$  under the inclusion map  $\varphi$ . If the induced metric  $h$  has Euclidean signature, the submanifold is called *Riemannian* or *spacelike*.

**Definition 5** Let  $\varphi : N \rightarrow M \in \mathcal{C}^\infty$ .  $\varphi$  is called an **immersion**, iff the **tangential map**  $T_p\varphi$  is injective  $\forall p \in N$ . Equally,  $\varphi$  is said to be an immersion iff  $\text{rank}(\varphi) = \dim(M)$ .

**Definition 6** Let  $\varphi : N \rightarrow M \in \mathcal{C}^\infty$ .  $\varphi$  is called a **topological embedding**, if  $\varphi$  is a homeomorphism<sup>1</sup> from  $N$  onto its image  $\varphi(N) \subseteq M$ . In other words,  $N$  is **diffeomorphic** to its image  $\varphi(N)$ .

**Definition 7** A **smooth embedding** is an injective immersion, which is a topological embedding in the sense of Definition 6.

*Remark:* Commonly a *smooth embedding* is used to specify the inclusion map. But notice that  $\forall p \in N$  an immersion  $\varphi$  is a *local embedding*, i.e.  $\forall p \in N$  there exists a neighbourhood  $U \subseteq N$  of  $p$  such that  $\varphi : U \rightarrow M$  is an embedding. Conversely, a local embedding is always an immersion.

Due to an inclusion map we are now able to determine the geometric structure of the abstract manifold  $N$ . Therefore, we need operations which allow us to relate geometric objects defined on  $M$  to objects defined on  $N$ . The essential operation yielding the claimed connection between  $N$  and  $M$  is the **pullback**.

**Definition 8** Let  $\varphi : N \rightarrow M \in \mathcal{C}^\infty$ . The **pullback** of any covariant tensor field  $\alpha$  of rank  $k$  in the tensor space over  $M$  by  $\varphi$  is defined as:

$$\varphi^*\alpha(p) = (T_p\varphi)^* \circ \alpha \circ \varphi(p). \quad (2.9)$$

For  $x_1, \dots, x_k \in \mathcal{T}_p(N)$  this writes in local coordinates:

$$\varphi^*\alpha(p)(x_1, \dots, x_k) = \alpha(\varphi(p))(T_p(x_1), \dots, T_p(x_k)). \quad (2.10)$$

*Remark:* Notice that if  $\varphi$  is a diffeomorphism the pullback map possess an inverse which is called the **pushforward**. In that case it turns out that  $\varphi^*$  is a linear isomorphism and  $(\varphi^*)^{-1} = (\varphi^{-1})^*$ .

---

<sup>1</sup>A homeomorphism between two topological spaces is a continuous function with a continuous inverse.

One of the most essential pullback operations is that of the spacetime metric  $\mathbf{g}$ , which defines the **induced metric**  $\mathbf{h}$  on the submanifold  $N$ :

$$\mathbf{h} := \varphi^* \mathbf{g}. \quad (2.11)$$

In local coordinates this writes as  $h_{ij} = \varphi^\mu_{,i} \varphi^\nu_{,j} g_{\mu\nu}$ .  $\mathbf{h}$  is also called the **first fundamental form** of  $N$ .

Since there exists an induced metric on the submanifold  $N$ , the question of an induced affine connection suggests itself: if  $N$  is a Riemannian submanifold of  $M$ , the Levi-Civita connection  $\nabla$  of  $M$  induces a connection  $D$  on  $N$ . More details on this can be found in the following section.

## 2.3 Geometry of Hypersurfaces

Since the previous section dealt with submanifolds in general, we now turn to a special class of submanifolds, namely **hypersurfaces**.

### 2.3.1 Definition

**Definition 9** *Let  $M$  be a  $n$ -dimensional semi-Riemannian manifold. A **hypersurface**  $H$  is a submanifold of dimension  $n-1$ . Equivalently, the codimension of  $H$  is 1.*

In the special case of General Relativity, where  $\dim(M) = 4$ , a hypersurface is a 3-dimensional submanifold. From now on, the considered submanifolds will be hypersurfaces if not mentioned otherwise. If  $\varphi$  is a smooth embedding, as given by Definition 7, it applies that

$$\begin{aligned} \varphi: \Sigma &\rightarrow M \\ \Sigma &\mapsto \varphi(\Sigma) \end{aligned} \quad (2.12)$$

defines a hypersurface  $\varphi(\Sigma)$  in  $M$ , where  $\Sigma$  is an abstract 3-dimensional manifold. Locally, a hypersurface can be defined as a set of points for which a (smooth) scalar field  $t$  defined on an open neighbourhood of  $M$  is constant:

$$\forall p \in M, \quad p \in \varphi(\Sigma) \iff t(p) = \text{const}. \quad (2.13)$$

For instance, consider a connected submanifold  $\varphi(\Sigma)$  of  $M$  with the trace topology of  $\mathbb{R}^3$ . Introducing local coordinates  $x^\mu = (t, x^1, x^2, x^3)$  in an open neighbourhood  $U$  of  $M$  and using Eq.(2.13), the hypersurface in  $M$  is defined by the coordinate condition  $t = \text{const}$ . Further, presuming that  $\{y^i\}$  are given coordinates on the abstract manifold  $\Sigma$ , an explicit form of the mapping can be obtained:

$$\begin{aligned} \varphi: \Sigma &\rightarrow M \\ \{y^i\} &\mapsto (x^\mu) \equiv (\varphi(t_0, y^i)), \end{aligned} \quad (2.14)$$

where we have chosen the constant to be  $t_0$ .

Due to the definition of an immersion (see Definition 5) any  $\mathbf{v} \in \mathcal{T}_p(\Sigma)$  can be mapped onto a vector  $\mathbf{w} \in \mathcal{T}_{\varphi(p)}(\varphi(\Sigma)) \subseteq \mathcal{T}_{\varphi(p)}(M)$ . Conversely, the pullback mapping (see Eq.(2.9)) allows us to identify covariant tensor fields on  $M$  with covariant tensor fields on the abstract manifold  $\Sigma$ . The pullback of the spacetime metric  $\mathbf{g}$  and its identification with the metric on the abstract manifold  $\Sigma$  has already been mentioned before (see Eq.(2.11)).

*Remark:* In general, so-called *globally hyperbolic* spacetimes are considered when developing a method to solve the Cauchy problem in General Relativity, since this is the most relevant case for the initial value formulation. A globally hyperbolic spacetime admits the existence of a special hypersurface, namely a *Cauchy surface*:

**Definition 10** *A subset of  $M$  is called a **Cauchy surface**, iff each timelike inextendible curve intersects it once and only once.*

But not all possible spacetimes admit a Cauchy surface. For instance, spacetimes with closed timelike curves are not globally hyperbolic. Nonetheless, there exist some results that spacetimes with closed timelike curves possess an initial value formulation [40], but causality arguments show, that it is not possible to prescribe *arbitrary* initial data on a spacelike hypersurface in such spacetimes.

If not mentioned otherwise, from now on we consider  $\Sigma$  to be a 3-dimensional **Riemannian manifold** such that  $\varphi(\Sigma) \subseteq M$ , where  $M$  is a 4-dimensional Lorentzian manifold and  $\varphi(\Sigma)$  a spacelike, i.e. Riemannian, hypersurface in  $M$ . As aforementioned (see Eq.(2.13)), it is possible to locally parametrise the hypersurface by any regular scalar field  $t$  such, that  $\varphi(\Sigma)$  represents an instant of time, i.e. it is locally a level surface of this scalar field. Physically spoken  $\varphi(\Sigma)$  is then a surface of simultaneity.

*Remark:* It is common to use an entire *foliation* of such hypersurfaces which covers  $M$  completely, but this is not necessary for our purposes. By now we have *one and only one* 3-dimensional Riemannian manifold  $\Sigma$  which is embedded in  $M$ .

### 2.3.2 Unit normal

Since  $\varphi(\Sigma)$  is a spacelike hypersurface in  $M$ , it admits a **timelike normal**. If the hypersurface is locally parametrised by a *regular* scalar field  $t$ , then the gradient 1-form defines the normal at a point  $p \in \varphi(\Sigma)$ :

$$\tilde{n}_\mu = \nabla_\mu t \quad (2.15)$$

It is common to choose the normal to be *normalised*, hence we refer to it as the **unit normal**  $\mathbf{n}$  by imposing the following *normalisation condition*:

$$g_{\mu\nu}n^\mu n^\nu = -1. \quad (2.16)$$

The unit normal  $\mathbf{n}$  points in a direction orthogonal to  $\varphi(\Sigma)$  in the sense that the pullback of the co-normal  $n_\mu$  vanishes:

$$\varphi^*n = 0. \quad (2.17)$$

This is the defining equation for the unit normal. In the following we refer to the above equation as the *orthonormality condition*. Its coordinate version writes as  $\varphi^*_i n_\mu = 0$ . We conclude, that the normal  $\mathbf{n}$  at the point  $p \in \varphi(\Sigma)$  is an element of  $\mathcal{T}_p^*\varphi(\Sigma)^\perp$ , which is the orthogonal complement of  $\mathcal{T}_p^*\varphi(\Sigma)$ . Consequently, the scalar product between any vector  $\mathbf{v}$  in  $\mathcal{T}_p\varphi(\Sigma)$  and the unit normal,  $g(n, v) = v^\mu n_\mu$ , vanishes identically.

Notice that the normal of a null hypersurface is tangent to it at the same time. Further, if  $\mathbf{n}$  is null it cannot be re-normalised. Thus, in this case and only in this case, there is no way of uniquely splitting any vector into a part tangential and a part normal to  $\varphi(\Sigma)$ . In contrast, if a *spacelike hypersurface* is considered, any vector  $x \in \mathcal{T}_p(M)$  can be uniquely split into parts tangential and normal to  $\varphi(\Sigma)$ :

$$x = \tan(x) + \text{nor}(x), \quad (2.18)$$

where  $\tan(x) \in \mathcal{T}_p\varphi(\Sigma)$  and  $\text{nor}(x) \in \mathcal{T}_p^*\varphi(\Sigma)^\perp$  are the appropriate *tangential and normal projections*. This is equivalent to the decomposition of the tangent space  $\mathcal{T}_p(M)$  itself.

### 2.3.3 Intrinsic curvature of $\varphi(\Sigma)$

However  $\Sigma$  is mapped onto  $M$ , it possesses an **intrinsic curvature** which encodes the information about whether  $\Sigma$  is flat or curved. Since the triple  $(\Sigma, h, D)$  describes an abstract manifold with Riemannian metric  $\mathbf{h}$  and Levi-Civita connection (or covariant derivative)  $\mathbf{D}$ , there exists a Riemann curvature tensor  ${}^3R$  associated with this connection, which represents the intrinsic curvature of  $\Sigma$ . Following the notation introduced in Sec. 2.1, the components of  ${}^3R$  are denoted by  ${}^3R^i_{\phantom{i}kjl}$ .

As already mentioned before, many books on differential geometry refer to the Ricci identity (see Eq.(2.5)) as the defining equation for the curvature tensor. Thus, one might define  ${}^3R$  by:

**Definition 11** (*Ricci identity*)

$$\forall \mathbf{v} \in \mathcal{T}_p(\Sigma), \quad (D_j D_l - D_l D_j)v^i = {}^3R^i_{\phantom{i}kjl}v^k. \quad (2.19)$$

The corresponding Ricci tensor  ${}^3Ric$  of  $\Sigma$  can be obtained by calculating the  $(1, 3)$  contraction of  ${}^3R$ :

$${}^3R_{ij} = {}^3R^k{}_{ikj}. \quad (2.20)$$

Analogous to the case of the entire spacetime (see Eq.(2.8)) the **scalar curvature** can be determined by taking the trace of the Ricci tensor with respect to the 3-metric  $\mathbf{h}$ :

$${}^3\mathcal{R} = h^{ij}{}^3R_{ij} = tr(R_{ij}) = {}^3R^i{}_i. \quad (2.21)$$

#### 2.3.4 Extrinsic curvature of $\varphi(\Sigma)$

In addition to the intrinsic curvature of  $\Sigma$  we discussed before, another type of curvature regarding the inclusion of the abstract manifold  $\Sigma$  in the ambient manifold  $M$  can be defined. We may call this one the **extrinsic curvature  $\mathbf{K}$**  of the hypersurface  $\varphi(\Sigma)$ . This special curvature is related to the *bending of  $\Sigma$*  in the spacetime manifold  $M$  under the embedding  $\varphi$ . For instance, assume that  $\Sigma$  is intrinsically flat, i.e.  ${}^3R$  vanishes. Now, take an embedding  $\varphi$  such, that the hypersurface  $\varphi(\Sigma)$  appears to be bent. This “bending” is mathematically examined the following: compare the normal defined at the point  $p_1 \in \varphi(\Sigma)$  with the normal at the point  $p_2 \in \varphi(\Sigma)$ ,  $p_1 \neq p_2$  by parallel transporting  $n(p_1)$  along tangential vectors to  $p_2$  w.r.t. the spacetime connection  $\nabla$ . If the parallel transported normal  $n(p_1)$  and the normal in  $p_2$  do not coincide, the hypersurface is bent in  $M$ . Hence, it appears to be curved in the ambient manifold, although it is still intrinsically flat. More precisely, the extrinsic curvature is defined as the following:

**Definition 12** *The **extrinsic curvature  $\mathbf{K}$**  of a hypersurface  $\varphi(\Sigma)$  is defined as the bilinear, symmetric map*

$$\begin{aligned} K : \mathcal{T}_p(\varphi(\Sigma)) \times \mathcal{T}_p(\varphi(\Sigma)) &\rightarrow \mathbb{R} \\ (\mathbf{v}, \mathbf{w}) &\mapsto -\mathbf{g}(\mathbf{v}, \nabla_{\mathbf{w}}\mathbf{n}). \end{aligned} \quad (2.22)$$

Its components are denoted by  $K_{\mu\nu}$ . Notice, that the extrinsic curvature is also often called the **second fundamental form** or the **shape operator**. In words, the extrinsic curvature describes the change of the normal direction as one moves along the embedded hypersurface.

Since  $\mathbf{K}$  is purely tangential to  $\varphi(\Sigma)$  per definition, i.e. its contraction with the unit normal  $\mathbf{n}$  vanishes, the pullback of  $\mathbf{K}$  under  $\varphi$  onto  $\Sigma$  itself encodes the same information as the object “living” in spacetime. Hence, instead of using  $K_{\mu\nu}$  one can equivalently deal with  $K_{ij}$ . We are here, and often in what follows, identifying the tangent space of  $\varphi(\Sigma)$  with the tangent space of  $\Sigma$  itself.

As aforementioned, the link between the extrinsic curvature and the unit

normal to the embedded hypersurface is rather illustrative. But in order to perform explicit calculations a formula, connecting the extrinsic curvature and unit normal rigorously, is needed. Therefore, we use the definition of the extrinsic curvature tensor Eq.(2.22) and evaluate it using any pair of vectors  $(v, w) \in \mathcal{T}_p(M) \times \mathcal{T}_p(M)^2$ . Since  $\mathbf{K}$  is a bilinear form defined for two vectors tangent to  $\varphi(\Sigma)$ , an operation called *projection* (see below) has to be used in order to extract only the tangential parts of the vectors. Hence, it follows directly:

$$K_{ij} = -\varphi_{;i}^\mu \varphi_{;j}^\nu \nabla_\mu n_\nu. \quad (2.23)$$

Notice, that the minus sign is a question of convention.

In order to obtain the associated *mean curvature*, we take the trace either with respect to the spacetime metric  $\mathbf{g}$  or to the spatial metric  $\mathbf{h}$ :

$$\mathcal{K} = g^{\mu\nu} K_{\mu\nu} = h^{ij} K_{ij} = K^i_i. \quad (2.24)$$

Taking the trace of Eq.(2.23) yields a simple relation between the trace of  $\mathbf{K}$  and the divergence of the unit normal:

$$\mathcal{K} = -\nabla_\mu n^\mu. \quad (2.25)$$

*Remark:* Although we have not yet mentioned an entire family of hypersurfaces, we have to make an important remark here. In the case of a so-called foliation (see below), the unit normal is defined on  $M$  and commonly referred to as the 4-velocity of some observer. Such an observer, called *Eulerian observer* in Sec. 2.4.2, can be defined by choosing a non-zero lapse and arbitrary shift in the language of Sec. 2.4.1. For example, choosing the lapse equal to one, corresponds to extending the unit normal  $n_\mu$  off  $\varphi(\Sigma)$  by requiring it to be geodesic. For Eulerian observers a useful relation can then be found:

$$\nabla_\mu n_\nu = -K_{\mu\nu} - a_\nu n_\mu, \quad (2.26)$$

where  $a_\lambda := n^\sigma \nabla_\sigma n_\lambda$  is the 4-acceleration of this observer. The comprehensive explanation can be found in [20]. Let us emphasise here that such a definition does not make sense in the case of a general slicing which we want to consider later. The explanation can be found in Sec. 2.4.

### 2.3.5 Projection onto $\Sigma$

We have seen in Sec. 2.3.2 that the tangent space of the ambient manifold  $(M, g)$  at each point  $p \in \varphi(\Sigma)$  can be decomposed uniquely into the direct sum of the space tangential to the spacelike hypersurface  $\varphi(\Sigma)$ <sup>3</sup> and its

---

<sup>2</sup>Remember that the tangent space of  $M$  at the point  $p$  can be decomposed uniquely in a component tangential to the  $\varphi(\Sigma)$  and one orthogonal to it.

<sup>3</sup>Remember that  $\mathcal{T}_p\varphi(\Sigma)$  is also the tangent space to  $\Sigma$  itself as  $\varphi$  is a diffeomorphism.

orthogonal complement. Hence, we are free to choose the subspace generated by the unit normal  $\mathbf{n}$  to span the orthogonal complement to  $\mathcal{T}_p(\Sigma)$ :

$$\mathcal{T}_p(M) = \mathcal{T}_p(\Sigma) \oplus \text{Vec}(\mathbf{n}), \quad (2.27)$$

where  $\text{Vec}(\mathbf{n}) \subseteq \mathcal{T}_p(\Sigma)^\perp$ . Equivalently, this means that any vector  $\mathbf{x}$  in  $\mathcal{T}_p(M)$  can be decomposed uniquely into components tangential and orthogonal to  $\varphi(\Sigma)$  via

$$x^\mu = x^\mu_{\parallel} + x^\mu_{\perp}, \quad \forall x \in \mathcal{T}_p(M). \quad (2.28)$$

The idea of a so-called **projection onto the hypersurface** is to “filter” the tangential component of a spacetime vector or its dual, a spacetime 1-form, or objects of higher rank. This tangential part can then be identified as an object in  $\mathcal{T}_p(\Sigma)$  due to the smooth embedding.

*Remark:* Since a smooth embedding is used,  $\Sigma$  is diffeomorphic to its image  $\varphi(\Sigma)$  and thus an inverse mapping can be found, which we call the **extension operator** (see below).

It appears to be natural to consider the *projection of covariant objects* rather than contravariant ones. Therefore, the projection coincides with the pullback mapping in order to reduce objects in  $\mathcal{T}_p^*(M)$  to objects in  $\mathcal{T}_p^*(\Sigma)$  (or products of these spaces):

**Definition 13** *The orthogonal projection onto  $\Sigma$  is defined as the pullback map*

$$\varphi^* \omega = \omega_{\parallel}, \quad \forall \omega \in \mathcal{T}_p^*(M), \quad (2.29)$$

where  $\omega_{\parallel}$  is an element of  $\mathcal{T}_p^*(\Sigma)$ . Using index notation this writes as

$$\varphi^\mu_{\phantom{\mu},i} \omega_\mu = \omega_i \in \mathcal{T}_p^*(\Sigma). \quad (2.30)$$

In words, the pullback of any element of the linear space of covariant tensors of rank  $k$  over  $M$  yields objects of the same rank in the linear space over  $\Sigma$  and thus is called the **orthogonal projection onto  $\Sigma$** . In a chosen basis its components are given by:

$$(\varphi^* \mathbf{T})_{\alpha_1 \dots \alpha_k} = \varphi^\mu_{\phantom{\mu},i_1} \dots \varphi^\mu_{\phantom{\mu},i_k} T_{\mu_1 \dots \mu_k} = T_{i_1 \dots i_k}. \quad (2.31)$$

As mentioned before, one might also extend a purely spatial object, i.e. an object defined on  $\Sigma$ , to an object on  $M$ . This can be done since the map  $\varphi$  is an embedding and thus provides an injective tangential map. Therefore, the **extension** appears to be a natural inverse to the orthogonal projection and is denoted by  $\gamma$ . In the easiest case it is a map

$$\gamma: \mathcal{T}_p^*(\Sigma) \rightarrow \mathcal{T}_p^*(M). \quad (2.32)$$

The extension map is also defined on the product spaces and therefore applies to covariant tensors of any rank. Let us illustrate this extension map in the case of the spatial metric  $\mathbf{h}$ , which is a symmetric bilinear form on  $\Sigma$ .  $\mathbf{h}$  can only be extended to a covariant tensor of rank 2 on  $M$  such that it satisfies Eq.(2.11). Hence, it can be expressed in terms of the spacetime metric  $\mathbf{g}$  and the dual to the unit normal  $\mathbf{n}$ :

$$h_{\mu\nu} = \gamma_\mu^i \gamma_\nu^j h_{ij} \equiv g_{\mu\nu} + n_\mu n_\nu. \quad (2.33)$$

Applying the pullback or orthogonal projection to Eq.(2.33) actually provides the original spatial metric  $h_{ij}$  on  $\Sigma$  as given by Eq.(2.11), exactly as demanded. The component version of the extension operation is

$$\gamma_\mu^i = g_{\mu\nu} \varphi_{,j}^\nu h^{ij}. \quad (2.34)$$

Similarly, we might use this extension to turn the extrinsic curvature tensor  $\mathbf{K}$ , originally defined as a bilinear form on  $\Sigma$ , into a bilinear form on  $M$ . Then, the extension is given by:

$$K_{\mu\nu} = \gamma_\mu^i \gamma_\nu^j K_{ij}. \quad (2.35)$$

Notice that  $K_{\mu\nu}$  is still tangential to  $\varphi(\Sigma)$  as it is defined as a symmetric bilinear form on  $\mathcal{T}_p^* \varphi(\Sigma) \times \mathcal{T}_p^* \varphi(\Sigma)$  by Eq.(2.22).

Let us adhere the main conclusions by now: the orthogonal projection of a tensor on  $M$  gives only its part tangential to  $\varphi(\Sigma)$  and the projection of any tensor normal to  $\varphi(\Sigma)$  vanishes identically.

Applying Eq.(2.31) to a tensor field  $\mathbf{T}$  tangential to  $\varphi(\Sigma)$  yields a direct translation between the covariant derivative with respect to the spacetime Levi-Civita connection  $\nabla$  and the covariant derivative of this tensor field w.r.t. the induced connection  $\mathbf{D}$  on the spacelike hypersurface, respectively on  $\Sigma$  itself.

$$\mathbf{D}\mathbf{T} = \varphi^* \nabla \mathbf{T}. \quad (2.36)$$

The component version writes similar to Eq.(2.31):

$$D_k T_{i_1 \dots i_p} = \varphi_{,k}^\mu \varphi_{,i_1}^{\alpha_1} \dots \varphi_{,i_p}^{\alpha_p} \nabla_\mu T_{\alpha_1 \dots \alpha_p} \quad (2.37)$$

*Remark:* The domain of  $\mathbf{T}$  on the right-hand side of Eq.(2.36) is only the hypersurface  $\varphi(\Sigma)$  and therefore it represents directly the 4-dimensional version of a spatial tensor field.

Further, making use of Eq.(2.36) yields an interesting and useful relation for the derivative of vector field  $\mathbf{v}$  along another vector field  $\mathbf{w}$ , when both



of them are tangent to  $\varphi(\Sigma)$ . According to Eq.(2.33) we find:

$$\begin{aligned}\mathbf{D}_{\mathbf{w}}\mathbf{v} &= (\mathbf{w} \cdot \mathbf{D})\mathbf{v} \\ &= \gamma[(\mathbf{w} \cdot \nabla)\mathbf{v}] \\ &= (\mathbf{w} \cdot \nabla)\mathbf{v} + \mathbf{n} \otimes \mathbf{n}(\mathbf{w} \cdot \nabla)\mathbf{v}.\end{aligned}\tag{2.38}$$

Proving Eq.(2.36) needs two steps: first, one has to show that the pullback of the spacetime connection induces a torsion-free connection on  $\Sigma$  which satisfies all required properties. This is done easily and hence not demonstrated here. Secondly, the connection applied to the spatial metric  $\mathbf{h}$  must vanish identically:

$$\begin{aligned}D_k h_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,k}^\sigma \nabla_\sigma h_{\mu\nu} \\ &= \varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,k}^\sigma (\nabla_\sigma g_{\mu\nu} + (\nabla_\sigma n_\mu) n_\nu + n_\mu \nabla_\sigma n_\nu) \\ &= 0,\end{aligned}\tag{2.39}$$

since the pullback of the normal 1-form and the covariant derivative of  $\mathbf{g}$  are zero.

According to the uniqueness of a torsion-free connection associated with a non-degenerate metric, we conclude that necessarily

$$\varphi^* \nabla = \mathbf{D}.\tag{2.40}$$

### 2.3.6 The Codazzi relation

So far, we have developed several relations between the spacetime objects and their counterparts belonging to  $\Sigma$ , like the relation between the two metrics. But since the geometric object encoding the forces of gravity is the Riemann curvature tensor, we are more than interested in deducing relations between the Riemannian of the 4-dimensional spacetime  $(M, g)$  and the 3-dimensional one of the hypersurface  $(\Sigma, h)$ . The following equations are the basis of the 3 + 1 version of General Relativity.

*Note:* As is customary, the calculations in this and in the following subsection assume that the unit normal  $\mathbf{n}$  is also defined off the hypersurface  $\varphi(\Sigma)$ . Such an extension can always be found, e.g. using an Eulerian observer. It turns out though, that the final result does not depend on such an extension.

Let us start with the easier equation to derive, the so-called **Codazzi relation**<sup>4</sup>. Therefore, we take the Ricci identity Eq.(2.19) and apply it to the unit normal vector  $\mathbf{n}$ :

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n_\sigma = 2 \nabla_{[\mu} \nabla_{\nu]} n_\sigma = {}^4 R_{\mu\nu\sigma}{}^\lambda n_\lambda,\tag{2.41}$$

---

<sup>4</sup>often referred to as Codazzi-Mainardi equations

where the square brackets denote the anti-symmetrisation operation. In order to establish a relation between the 4- and the 3-dimensional objects, one has to project the above equation onto the considered hypersurface as demonstrated in the previous subsection. Hence, applying the orthogonal projection yields

$$\varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu {}^4R_{\mu\nu\sigma}{}^\lambda n_\lambda = 2\varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu (\nabla_{[\mu} \nabla_{\nu]} n_\sigma). \quad (2.42)$$

Now, taking into account Eq.(2.26), it follows:

$$\begin{aligned} 2\varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu \nabla_{[\mu} \nabla_{\nu]} n_\sigma &= \varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu \nabla_\mu (-K_{\nu\sigma} - a_\sigma n_\nu) - \\ &\quad - \varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu \nabla_\nu (-K_{\mu\sigma} - a_\sigma n_\mu) \\ &= \varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu (-\nabla_\mu K_{\nu\sigma} - a_\sigma \nabla_\mu n_\nu - n_\nu \nabla_\mu a_\sigma) - \\ &\quad - \varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu (-\nabla_\nu K_{\mu\sigma} - a_\sigma \nabla_\nu n_\mu - n_\mu \nabla_\nu a_\sigma) \\ &= -D_i K_{jk} + D_j K_{ik} \end{aligned} \quad (2.43)$$

where we have used the orthonormality relation Eq.(2.17) to obtain the last line. Thus, we conclude

$$\varphi_{,k}^\sigma \varphi_{,i}^\mu \varphi_{,j}^\nu {}^4R_{\mu\nu\sigma}{}^\lambda n_\lambda = D_j K_{ik} - D_i K_{jk}. \quad (2.44)$$

This is the famous **Codazzi equation**, relating the 4-dimensional Riemann curvature tensor the the spatial covariant derivative of the extrinsic curvature.

Taking the (3,4)-contraction of Eq.(2.44) leads to the *contracted Codazzi relation*:

$$\varphi_{,i}^\mu n^{\nu 4} R_{\mu\nu} = D_i \mathcal{K} - D_k K_i{}^k, \quad (2.45)$$

where we have used

$$\gamma_\lambda^k \varphi_{,k}^\sigma = h_\lambda^\sigma = \delta_\lambda^\sigma + n^\sigma n_\lambda. \quad (2.46)$$

### 2.3.7 The Gauß relation

The second important relation is the so-called **Gauß relation**: as given by Eq.(2.19), the Riemann tensor of the  $\Sigma$  measures the lack of commutation of two successive covariant derivatives with respect to the connection  $D$ . Using the definitions given in the previous sections and applying the relation between the spacetime connection and the induced connection on  $\Sigma$  (see Eq.(2.36)), we obtain:

$$D_i D_j v^k = \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k \nabla_\mu (D_\nu v^\sigma), \quad \forall v \in \mathcal{T}_p(\Sigma). \quad (2.47)$$

Notice that  $v^\sigma$  is the 4-dimensional version of the 3-vector tangent to  $\Sigma$  and thus also tangent to  $\varphi(\Sigma)$ , whereas  $D_\nu$  denotes the metric connection on the

embedded hypersurface  $\varphi(\Sigma)$ . Applying the definitions of Sec. 2.3.5 and using Eq.(2.36) again, we gain the following explicit form of these extensions:

$$\begin{aligned} D_\nu v^\sigma &= \gamma_\nu^m \varphi_{,n}^\sigma D_m v^n \\ &= \gamma_\nu^m \varphi_{,n}^\sigma \varphi_{,m}^\alpha \gamma_\beta^n \nabla_\alpha v^\beta \\ &= h^\alpha_\nu h^\sigma_\beta \nabla_\alpha v^\beta. \end{aligned} \quad (2.48)$$

Inserting this intermediate result into Eq.(2.47), we obtain altogether that

$$D_i D_j v^k = \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k \nabla_\mu (h^\alpha_\nu h^\sigma_\beta \nabla_\alpha v^\beta). \quad (2.49)$$

Using Eq.(2.46) and  $\nabla_\mu (\delta^\alpha_\beta + n^\alpha n_\beta) = n_\beta \nabla_\mu n^\alpha + n^\alpha \nabla_\mu n_\beta$  yields:

$$\begin{aligned} D_i D_j v^k &= - K_{ij} \gamma_\sigma^k n^\lambda \nabla_\lambda v^\sigma + \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k \nabla_\mu \nabla_\nu v^\sigma - \\ &- \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k K_\mu^\sigma K_{\nu\lambda} v^\lambda. \end{aligned} \quad (2.50)$$

Applying the same procedure to the second term in the Ricci identity, which is equivalent to the permutation of the indices  $i$  and  $j$ , as well as using that  $\mathbf{v}$  and  $\mathbf{K}$  are tangent to  $\Sigma$ , we acquire the following result:

$$\begin{aligned} (D_i D_j - D_j D_i) v^k &= (K_j^k K_{im} - K_i^k K_{jm}) v^m + \\ &+ \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k \nabla_\mu \nabla_\nu v^\sigma - \varphi_{,i}^\mu \varphi_{,j}^\nu \gamma_\sigma^k \nabla_\nu \nabla_\mu v^\sigma. \end{aligned} \quad (2.51)$$

Comparing the last two terms of the equation above with Eq.(2.5) and taking into account the right-hand side of Eq.(2.19), the very final result reads as:

$$\varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,m}^\lambda \gamma_\sigma^k R^\sigma_{\lambda\mu\nu} v^m = {}^3R^k_{mij} v^m + (K_j^k K_{im} - K_i^k K_{jm}) v^m. \quad (2.52)$$

Due to the presence of the projections and the fact that  ${}^3R$  and  $\mathbf{K}$  are tangent to  $\Sigma$ , the above result is even valid for any vector  $\mathbf{v} \in \mathcal{T}_p(M)$  and thus we conclude:

$$\varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,m}^\lambda \gamma_\sigma^k R^\sigma_{\lambda\mu\nu} = {}^3R^k_{mij} + (K_j^k K_{im} - K_i^k K_{jm}). \quad (2.53)$$

This is the famous **Gauß relation** which was originally found for 2-dimensional surfaces embedded in the Euclidean  $\mathbb{R}^3$ .

Applying the  $(1,2)$ -contraction to Eq.(2.53) and using the symmetry relations of the Riemann tensor we obtain the *contracted Gauß relation*:

$$\varphi_{,i}^\mu \varphi_{,j}^\nu {}^4R_{\mu\nu} + \varphi_{,i}^\mu \varphi_{,j}^\nu {}^4R^\sigma_{\mu\lambda\nu} n^\lambda n_\sigma = {}^3R_{ij} + \mathcal{K} K_{ij} - K_{im} K^m_j \quad (2.54)$$

Now, taking the trace of the above equation with respect to the spatial metric yields the *scalar Gauß equation*, which is a generalisation of Gauß famous' *theorem egregium*:

$${}^4\mathcal{R} + 2{}^4R_{\mu\nu} n^\mu n^\nu = {}^3\mathcal{R} + K^2 - K_{ij} K^{ij}. \quad (2.55)$$

It directly relates the intrinsic curvature of  $\Sigma$  to its extrinsic curvature and the intrinsic curvature of the ambient manifold  $M$  represented by the Ricci scalar and the Ricci tensor.

The Gauß and the Codazzi relation constitute crucially to the basis of reformulating Einstein's equations in the  $3 + 1$  version since they provide the essential connection between the 4- and the 3-curvature.

### 2.3.8 Examples of Hypersurfaces

In order to illustrate the aforementioned definitions and concepts, we use a space which we are very familiar with, namely  $\mathbb{R}^3$  equipped with the standard Euclidean metric  $\mathbf{g}$  of signature  $(+, +, +)$ , which is a Riemannian metric instead of a Lorentzian one. Thus, the ambient manifold  $(M, g)$  is of dimension 3 and any hypersurface  $\varphi(\Sigma)$  therein is a surface by the common means of the word “surface”, i.e. a 2-dimensional submanifold. We present the three most common surfaces in  $\mathbb{R}^3$ : a plane, a cylinder and a sphere. A more detailed presentation can be found in [20].

#### Example 1: a plane

For our first illustration let us consider  $\Sigma$  to be a simple *plane*. Further we assume a chart on  $M = \mathbb{R}^3$  such, that Cartesian coordinates are given:  $(X^i) = (x, y, z)$ . The surface  $\varphi(\Sigma)$  shall be a level surface of the coordinate  $z$ , e.g.  $\varphi(\Sigma)$  shall be the plane given by  $z = 0$ . The tuple  $(y^i) = (x, y)$  constitutes an appropriate coordinate system on the abstract surface  $\Sigma$ . The induced metric then takes the standard form  $h = \text{diag}(+1, +1)$ . Obviously, this metric is flat, i.e.  ${}^2R$  vanishes. The components of the unit normal are  $n^i = (0, 0, 1)$  with respect to the Cartesian chart on  $M$ . One sees immediately that the gradient of the unit normal vanishes. As a consequence, the extrinsic curvature vanishes as well. Hence, a plane in  $\mathbb{R}^3$  is not only intrinsically flat, also its extrinsic curvature is zero.

#### Example 2: a cylinder

For the second example let us consider the surface  $\varphi(\Sigma)$  to be a very well-known geometric figure, namely a *cylinder* which is defined by the equation  $t := \rho - R = 0$ , where  $R < 0$  is the radius of the cylinder and  $\rho = \sqrt{x^2 + y^2}$  the standard radial coordinate, while  $\Sigma$  is again a simple plane. Introducing standard cylindrical coordinates  $(X^i) = (\rho, \phi, z)$  on  $M$ , such that  $\phi \in [0, 2\pi)$ ,  $x = R \cos \phi$  and  $y = R \sin \phi$ , also gives an appropriate coordinate system  $(y^i) = (\phi, z)$  on the surface respectively on  $\Sigma$  itself. The components of the induced metric in this coordinate system are given by

$$h_{ij}dx^i dx^j = R^2 d\phi^2 + dz^2. \quad (2.56)$$

Due to the appearance of polar coordinates, one might be tempted to think of this as a non-flat metric. But it actually is as demanded for a plane, which can be seen easily using a coordinate transformation like  $z \mapsto R\tilde{z}$ . In order to evaluate the extrinsic curvature, one has to calculate the unit normal explicitly, which writes with respect to the cylindrical coordinates as

$$n^i = \left( \frac{R \cos \phi}{\rho}, \frac{R \sin \phi}{\rho}, 0 \right). \quad (2.57)$$

Transforming the above result into Cartesian coordinates and calculating the gradient of it gives the extrinsic curvature. Taking the trace yields the expected final result:

$$\mathcal{K} = -\frac{1}{R}. \quad (2.58)$$

Hence, though  $\Sigma$  is an intrinsically flat plane, its immersive image, the cylinder, has an extrinsic curvature.

### Example 3: a sphere

In the third and last example we consider  $\Sigma$  to be a sphere of radius  $R$  and thus  $\varphi(\Sigma)$  represents a *sphere* embedded in  $\mathbb{R}^3$ . It is defined by  $t := r - R = 0$ , where  $r = \sqrt{x^2 + y^2 + z^2}$  is the common Cartesian radial coordinate. In order to use an appropriate coordinate system, we introduce the well-known spherical coordinates  $(X^i) = (r, \theta, \phi)$  on  $M$ . This choice constitutes the coordinate system  $(y^i) = (\theta, \phi)$  on  $\Sigma$  which obviously is an  $S^2 \subseteq \mathbb{R}^3$ . The components of the extended spatial metric  $\mathbf{h}$  on  $\varphi(\Sigma)$  are then given by

$$h_{ij}dx^i dx^j = R^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.59)$$

Notice that this metric is not flat compared to the ones of the examples before. Hence, the Riemann tensor of  $(\Sigma, h)$  does not vanish.

Applying the same procedure as before, we can calculate the unit normal of the embedded sphere and eventually its extrinsic curvature, which turns out to be

$$\mathcal{K} = -\frac{2}{R}. \quad (2.60)$$

Hence, we have an example which is neither intrinsically nor extrinsically flat. Both types of curvature do not vanish in this case.

These examples show that hypersurfaces with both types of curvature vanishing (plane) exist as well as hypersurfaces where only one curvature, in the cylindrical case the intrinsic one, vanishes but that there are also hypersurfaces with both curvatures not vanishing as seen in the spherical case. Remembering the Gauß equation (Eq.2.53), we have already seen that the extrinsic curvature is not fully independent from the intrinsic one. Further details can be found in [20].

## 2.4 Kinematics of Hypersurfaces

Until now we have only considered a *single* set  $\Sigma$  which is included into a Lorentzian manifold  $(M, g)$  via a smooth embedding  $\varphi$  such, that the originating submanifold is a spacelike, i.e. Riemannian, hypersurface  $\varphi(\Sigma)$  in  $M$ . But, according to the idea of an initial value formulation, we are interested in deducing the *kinematic evolution* of certain physically relevant quantities or even gravity itself.

In order to say something about the propagation of a point  $p$  originally contained in this one hypersurface, we have to introduce a *kinematic concept*. This is done either by considering a family of spacelike hypersurfaces, a so-called **foliation**, as commonly done or, as we prefer to do, by choosing a *1-parameter family of embeddings* or *slicing*. In the general case of a slicing, the intersection of adjacent hypersurfaces is allowed, whilst a foliation demands that hypersurfaces do not intersect. Commonly, a regular foliation is used in order to derive the  $3 + 1$  form of field equations but as we want to keep it as general as possible, we prefer a slicing instead though one has to be more careful with the derivation.

*Remark:* The aforementioned concept of a general slicing requires at least two important remarks. First, since we allow intersecting hypersurfaces in  $M$ , one might suppose that the unit normal is not well-defined. This is true in the sense, that  $\mathbf{n}$  is *not a vector field on  $M$*  anymore. But every hypersurface has a normal being well-defined in each point of the hypersurface although the adjacent hypersurface has a different unit normal vector field. Therefore we conclude that, mathematically spoken, in the case of a slicing the normal vector field is defined only along the embedding  $\varphi$ . Notice, that these normal vector fields coincide only in the case of a foliation such, that the unit normal vector field is again a vector field on  $M$ . Secondly, if a point  $p$  is at the same element of a hypersurface and the adjacent one as well, this only means that  $p$  is *not* propagated along the change of the family parameter.

Before turning to some important kinematic quantities, let us make the above concept more precise:

**Definition 14** *Let  $\Sigma$  be a topological set and  $\varphi_\lambda$  be a 1-parameter family of embeddings  $\varphi$  labelled by the parameter  $\lambda \in \mathbb{R}$ . A single **slice** is the image of the set  $\Sigma$  under the embedding  $\varphi$  at parameter value  $\lambda$ .*

Notice that the consideration of a family of embeddings is equivalent to operating on the Cartesian cross-product  $\mathbb{R} \times \Sigma$ . Hence, if  $\{y^i\}$  provides a local coordinate system on  $\Sigma$ , the coordinates  $(\lambda, y^i)$  are appropriate to

label a point  $q \in \mathbb{R} \times \Sigma$ . In local coordinates the map then writes as:

$$\begin{aligned} \varphi_\lambda : \quad \mathbb{R} \times \Sigma &\rightarrow M \\ (\lambda, y^i) &\mapsto (x^\mu) = \varphi(t, y^i). \end{aligned} \quad (2.61)$$

Considering again the important case of the unit normal we see immediately, that  $\mathbf{n}$  depends on  $t$  and  $y^i$  since it is only defined along the embedding and thus *not* a vector field on  $M$ . Hence, we write

$$\mathbf{n} = \mathbf{n}(\lambda, y^i). \quad (2.62)$$

*Remark:* If and only if we would choose a regular foliation instead, the unit normal would be a global vector field on  $M$  depending on  $x^\mu$ .

Since we are interested in the *time evolution*, it appears to be natural to consider the family parameter  $\lambda$  to be a *coordinate time*. Hence, from now on we use  $t$  instead of  $\lambda$  to label the hypersurfaces. With these conceptual assumptions we are now well-equipped to focus on the kinematics of points  $q \in \Sigma$  under the embeddings  $(\varphi_t)_{t \in \mathbb{R}}$ .

#### 2.4.1 Time evolution vector field

Originally, we started with only one topological set  $\Sigma$  and an embedding  $\varphi$  such that the originating submanifold is a spacelike hypersurface in  $M$ . In order to establish a kinematic description of points in  $\Sigma$ , we have already mentioned the concept of a slicing but we have not yet made clear how the slicing respectively the parameter family of embeddings is generated. This is the goal of this subsection.

Starting with a single hypersurface  $\varphi(\Sigma)$  an entire family of embeddings  $\varphi_t$  is generated by a vector field  $\xi$ . In contrast to the unit normal vector field, this **time evolution vector field** is really a vector field on  $M$ , although it is only defined in an open neighbourhood of a hypersurface. If we choose coordinates  $\{x^\mu\}$  on the domain of  $\xi$ , the time evolution vector field depends on these coordinates. Hence, we have  $\xi = \xi(x)$ . Further, when considering the *integral curves* of the vector field  $\xi$ , it becomes clear that it is the *generator of a family of embeddings*:

$$\xi^\mu(\varphi) = \dot{\varphi}^\mu, \quad (2.63)$$

where the “dot” represents the derivation with respect to the family parameter  $t$ . This obviously defines a system of *autonomic ordinary differential equations*. Naturally, an appropriate *initial condition* for this ODE system is

$$\varphi^\mu(0, y^i) = \varphi^\mu(y^i). \quad (2.64)$$

According to Eq.(2.18) the vector field  $\xi$  can be decomposed uniquely into its parts normal and tangential to *one* hypersurface  $\varphi_t(\Sigma)$ <sup>5</sup> by

$$\xi^\mu(\varphi(t, y)) = N(t, y)n^\mu(t, y) + \beta^\mu(t, y). \quad (2.65)$$

This is a completely general decomposition of the vector field  $\xi(x)$ . The scalar function  $N(t, y)$  is referred to as the **lapse function** and the tangential part  $\beta$  is the so-called **shift vector**. Notice that the shift is purely spatial, hence  $\beta^\mu = \varphi_{,j}^\mu \beta^j \in \mathcal{T}_p(\Sigma)$ . We have chosen this specific decomposition, since the parameter family shall allow us describe the kinematics of a all points in  $\Sigma$ .  $\xi$  can then be interpreted as the “flow of time” throughout the spacetime  $M$ . Any point  $p \in \varphi_t(\Sigma)$  can now be “transported” along this time evolution vector to an adjacent surface  $\varphi_{t+\Delta t}(\Sigma)$ .

*Remark:* Commonly, the lapse function is restricted to be strictly greater zero, i.e.  $N > 0$ . In a slicing the lapse can take any value, in particular it can vanish at certain points.

*Remark:* Notice that the vector field  $\xi(x)$  can describe different situations due to its explicit decomposition. If  $\xi$  vanishes at the point  $p \in \varphi_t(\Sigma)$ , it means that  $p$  is stuck in time and space. On the other hand, if only the lapse is zero but the shift not, this is equivalent to a reparametrisation of the hypersurface. And finally, if the shift vanishes, then the point is propagated in time but not along the spatial directions.

This effect of moving forward in time leads to the change of the spatial metric  $\mathbf{h}$  on the abstract 3-manifold  $\Sigma$ . Hence, it is obvious to interpret the ambient manifold  $(M, g)$  as the time development of a Riemannian metric  $\mathbf{h}$  on a 3-dimensional manifold. This clearly suggests that at least one important dynamical variable in the  $3 + 1$  formalism of General Relativity is the spatial metric [42]. Thus, regarding the initial value formulation of General Relativity, we expect appropriate initial data to consist of  $\Sigma$ , the spatial metric (the dynamical variable) and its “time derivative”, which is the second dynamical variable. The motion under  $\xi$  does not only lead to an evolution equation for the spatial metric, but as we see later it also leads to an evolution for the extrinsic curvature. Due to Einstein’s equations it turns out that these two variables together, the spatial metric and the extrinsic curvature, are subject to a closed system of time evolution equations, which is crucial in the context of the initial value formulation of General Relativity (see *Chapter 5*).

On the abstract 3-manifold  $\Sigma$  respectively on the Cartesian product space  $\mathbb{R} \times \Sigma$  the time derivative of a tensor defined on it is nothing else

---

<sup>5</sup>When considering only one hypersurface,  $t$  is fixed.



but the common and expected one: its *partial derivative* with respect to the parameter of the family of embeddings, i.e. it is simply  $\partial_t$  which we denote by a “dot”. But how can this be translated into an appropriate derivative of tensors in  $M$  defined on  $\varphi_t(\Sigma)$ ? Regarding Eq.(2.65) and its interpretation below, the natural time derivative in  $M$  must be related to the time evolution vector  $\xi$  somehow. Again, thinking of a single point  $p$  in  $\varphi_t(\Sigma)$  and not a covariant tensor field, its natural propagation is determined by the *flow of the vector field*  $\xi$ . Hence, each point “flows” along the integral curves of  $\xi$  into the adjacent surface. In the context of a point, this is often referred to as *Lie dragging* along the time evolution vector  $\xi$ . A detailed explanation of the Lie dragging and the **Lie derivative** is given in *Appendix A*. Since this operation is also appropriate to tensor fields, the Lie dragging of a tensor field defined on  $\varphi_t(\Sigma)$  along  $\xi$  results in a tensor field on  $\varphi_{t+\Delta t}(\Sigma)$ . Thus, the time derivative of a tensor  $\mathbf{T}$  on  $\Sigma$  is given by the pullback of the Lie derivative of the counterpart of this tensor in  $M$  along the embedding  $\varphi$ :

$$\partial_t T_{i_1 \dots i_n} = \varphi_{,i_1}^{\mu_1} \dots \varphi_{,i_n}^{\mu_n} \mathcal{L}_\xi T_{\mu_1 \dots \mu_n}. \quad (2.66)$$

We also denote the partial derivative of a tensor  $\mathbf{T}$  with respect to the family parameter  $t \in \mathbb{R}$  by the conventional “dot”  $\dot{T}$ .

Since the above equation is valid for any tensor field on  $M$  tangent to  $\varphi_t(\Sigma)$ , it applies particularly to the 3-metric  $\mathbf{h}$  yielding the following essential relation:

$$\dot{h}_{ij} = \varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\xi g_{\mu\nu}. \quad (2.67)$$

But what about a vector defined only along the embedding and *not* tangent to the hypersurface? What is the time derivative of the unit normal  $\mathbf{n}(t, y^i)$  for example? Therefore, we differentiate Eq.(2.17) with respect to the time parameter  $t$ , yielding

$$\dot{\varphi}_{,i}^\mu n_\mu + \varphi_{,i}^\mu \dot{n}_\mu = 0. \quad (2.68)$$

Starting with the above equation, one of the most essential relations is found. The explicit, cumbersome derivation can be found in *Appendix B* but the important result is mentioned here:

$$\dot{n}^\mu + \Gamma_{\nu\lambda}^\mu n^\nu \dot{\varphi}^\lambda = \varphi_{,i}^\mu (D^i N - K^i_j \beta^j). \quad (2.69)$$

Equivalently, we sometimes write

$$\xi^\lambda \nabla_\lambda n^\mu = \varphi_{,i}^\mu (D^i N - K^i_j \beta^j). \quad (2.70)$$

Hence, we can make this essential conclusion:

*If an arbitrary lapse  $N$  is considered, the time derivative of tangent and orthogonal objects is still defined.*

As mentioned before, the vector field  $\xi$  can describe various different situation. Naturally, this does have an effect on Eq.(2.69). In the case of a vanishing shift ( $\beta = 0$ ), i.e. a point is not moved along the spatial directions but only along the direction orthogonal to the initial hypersurface. Then Eq.(2.70) reads the following way:

$$N n^\lambda \nabla_\lambda n^\mu = \partial_t \bigg|_{\beta=0} n^\mu \equiv \varphi_{,i}^\mu D^i N. \quad (2.71)$$

On the other hand, if the lapse vanishes, i.e.  $N = 0$ , this yields

$$\varphi_{,j}^\lambda \beta^j \nabla_\lambda n^\mu = -\varphi_{,i}^\mu K_j^i \beta^j. \quad (2.72)$$

We conclude, that the time derivative splits into two parts: one dependent on the lapse  $N$  and the other one dependent on the shift  $\beta$ .

Let us mention an important observation concerning the vector field  $\xi$ . As seen before, it is the generator of the 1-parameter family of spacelike hypersurfaces and since we do not impose any constraints on it, it generates a completely general slicing. *BUT*:

*If the time evolution vector field is transversal to the hypersurface in every point  $p$  it generates a regular foliation.*

*If the time evolution vector is **not** transversal to the hypersurface in every point  $p$ , it generates a general slicing.*

## 2.4.2 Slicing vs. foliation

So far, we have derived the kinematic concept in a completely general setting. But let us use this section to highlight the main and essential differences between this general context and the commonly used foliation setting.

When using “only” a regular foliation, i.e.  $N > 0$ , it makes sense to introduce *observers* since we are working in a relativistic context. In Sec. 2.3 we have introduced the unit normal of a single hypersurface. In the case of a foliation the normal  $\mathbf{n}$  is a well-defined vector field on  $M$ . Since it is timelike and even normalised, it can be regarded as the *4-velocity* of a certain observer. The integral curves of the normal vector field are timelike and non-intersecting and hence represent the *wordlines* of this specific observer, normally called an *Eulerian* or *fiducial observer*. A hypersurface then represents the *set of simultaneity* for this observer. Such observers do not exist in a general slicing, since the unit normal vector field is not defined on the manifold itself. Hence, no congruence with  $\mathbf{n}$  as tangent vector is

generated.

Further, the lapse function can be interpreted as the proportionality factor relating the coordinate time  $t$  to the proper time of the Eulerian observer, as originally introduced by Wheeler. Such an interpretation cannot be given in the slicing context since the label “observer” does not have a reasonable meaning. But if a regular foliation is considered, a 4-*acceleration* can be associated with the fiducial observer. It is then defined by

$$a^\mu = n^\sigma \nabla_\sigma n^\mu. \quad (2.73)$$

Note that the 4-acceleration is orthogonal to the unit normal  $\mathbf{n}$  and thus an element of  $\mathcal{T}\varphi_t(\Sigma)$ . Hence, the extended 4-acceleration and its pullback are identical. It follows immediately from Eq.(2.71) that the lapse and the 4-acceleration are related by

$$Na_i = D_i N. \quad (2.74)$$

But an expression like the 4-acceleration does not make any sense at all when a general slicing is considered: in this case, the unit normal is only defined along the embedding, whereas at least a local definition is required in order to establish the validity of Eq.(2.73). But if a foliation is considered, a very useful relation between this 4-acceleration and the extrinsic curvature can be established (see Eq.(2.26)).

Commonly, the lapse function is restricted to be  $N > 0$ . In our more general setting, where intersecting surfaces are allowed, the lapse can take any value, in particular it can become zero at certain points. A point, where the lapse function vanishes (intersection point) can be seen as an event where the time “freezes in”. This does not happen in a foliation although it is of physical relevance: for instance, consider the Penrose diagram of a Schwarzschild spacetime. There might be interest in a slicing such, that both horizon crossings (lapse changes sign here) and the bifurcation sphere (lapse vanishes) are included. Normally, this is avoided by choosing a different slicing, but it can be of interest when certain constraints are imposed on the extrinsic curvature as well as in cosmological contexts [4].

We already mentioned several time that the unit normal vector field is only defined along the embedding  $\varphi_t$  when a general slicing is considered as opposed to the foliation setting, where it is a vector field on  $M$ . To be mathematically precise, we give a rigorous definition of the unit normal vector field for a slicing:

**Definition 15** *Let  $M$  be a 4-dimensional Lorentzian manifold and  $\varphi_t$  a 1-parameter family of smooth embeddings. The **pullback bundle** of  $M$  w.r.t.*

the map  $(t, y^i) \in \mathbb{R} \times \Sigma \mapsto x = \varphi_t(y^i) \in M$  is then defined as

$$\varphi^*(\mathcal{T}M) := \bigsqcup_p \{p\} \times \mathcal{T}_{\varphi_t(p)}M. \quad (2.75)$$

Hence, it is the disjoint union of all  $p \in \mathbb{R} \times \Sigma$  and the tangent spaces at  $\varphi_t(p)$ .

Let us now consider two points  $q_1$  and  $q_2$  on the product space  $\mathbb{R} \times \Sigma$ . We choose a chart on this space such, that the points are given by the following coordinates:  $q_1 = (t_1, y_1^i)$  and  $q_2 = (t_2, y_2^i)$ . A slicing allows that these originally different point are mapped onto the same point  $q$  under the embedding. Thus, let us consider that  $q$  is such an intersection point between two adjacent hypersurfaces  $\varphi_1(\Sigma)$  and  $\varphi_2(\Sigma)$ . Choosing coordinates on  $M$  the point is then at the same represented by  $(t_1, \varphi_1(y_1^i))$  and by  $(t_2, \varphi_2(y_2^i))$ . This is already a main difference to the foliation setting, where this cannot happen at all. Additionally, two different tangent spaces and unit normals can be found at this point: one with respect to  $\varphi_1(\Sigma)$  and one with respect to the other hypersurface. Hence, it is obvious that  $\mathbf{n}$  cannot be a vector field on  $M$ . But, if we consider the *pullback bundle*, the unit normal  $\mathbf{n}$  is well-defined as a *section* of the pullback bundle and thus represents a vector field on  $\mathbb{R} \times \Sigma$ . Due to this definition the degenerate point in  $M$  is identified correctly with the two different original points and further each point is associated with the unique and correct unit normal. We conclude, that some vector fields formerly defined only along the embedding are well-defined as sections of the pullback bundle of  $M$  but not as sections of the tangent bundle of  $M$ .

### 2.4.3 Evolution of the spatial metric

Regarding the considerations in the last sections, we have already seen that one essential dynamical variable in the initial value formulation of General Relativity is the spatial metric itself. Before deriving its evolution equations, let us take a look at a simple example to illustrate this:

*Consider a spherically symmetric solution of the vacuum Einstein field equations. By the Birkhoff theorem we know that the most general such solution is the Schwarzschild solution. Physically it describes the gravitational field outside a non-rotating, spherical mass distribution. To make it more precise consider a Schwarzschild black hole observed at a time  $t = 0$ . Hence,  $\mathbf{h}(t = 0) \in \mathcal{T}_p^*(\Sigma) \times \mathcal{T}_p^*(\Sigma)$  has to be chosen such that it describes a Schwarzschild black hole. Since we are interested in the dynamical evolution of the spacetime including a Schwarzschild black hole, it is evident that the time evolution of the metric yields the adequate description of the physical system as the metric is the characterising paramter.*

The above example shall visualise why the spatial metric  $\mathbf{h}$  is a basic dynamical variable. We now turn to the explicit derivation of the time evolution equations for the 3-metric. The starting point is Eq.(2.67). Therefore we get:

$$\begin{aligned}
\dot{h}_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\xi g_{\mu\nu} \\
&= \varphi_{,i}^\mu \varphi_{,j}^\nu (\xi^\rho \nabla_\rho g_{\mu\nu} + g_{\mu\rho} \nabla_\nu \xi^\rho + g_{\rho\nu} \nabla_\mu \xi^\rho) \\
&= \varphi_{,i}^\mu \varphi_{,j}^\nu (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \\
&= \varphi_{,i}^\mu \varphi_{,j}^\nu (n_\nu \nabla_\mu N + N \nabla_\mu n_\nu + \nabla_\mu \beta_\nu + n_\mu \nabla_\nu N + N \nabla_\nu n_\mu + \nabla_\nu \beta_\mu) \\
&= \varphi_{,i}^\mu \varphi_{,j}^\nu (-2N K_{\mu\nu} + \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu) \\
&= -2N K_{ij} + D_i \beta_j + D_j \beta_i \\
&= -2N K_{ij} + \mathcal{L}_\beta h_{ij},
\end{aligned} \tag{2.76}$$

where we have used the definition of the Lie derivative (see *Appendix A*), the explicit form of the time evolution vector  $\xi$  given by Eq.(2.65), the relation between the gradient of  $\mathbf{n}$  and the extrinsic curvature  $\mathbf{K}$  as stated in Eq.(2.23), as well as the vanishing of the covariant derivative of the spacetime metric respectively the spatial metric. Note that all derivatives in the fourth line on the right make sense even for general slicings, since all derivatives are purely tangential. Hence, we conclude that the **evolution of the spatial metric  $\mathbf{h}$**  is given by

$$(\partial_t - \mathcal{L}_\beta)h_{ij} = -2N K_{ij}. \tag{2.77}$$

*Remark 1:* We will see below, that the time evolution of the *dynamical variable*  $\mathbf{h}$  can be expressed in terms of the extrinsic curvature. Thus,  $\mathbf{K}$  is the second quantity we need to construct valid initial data for the general relativistic Cauchy problem. Hence, appropriate initial data shall consist of the triple  $(\Sigma, \mathbf{h}, \mathbf{K})$ .

*Remark 2:* We have not used the Einstein equations until now. All concepts taken into account at the moment are purely geometrical. In order to rewrite the field equations for General Relativity as an initial value problem, we still have to find the evolution equations for the extrinsic curvature  $\mathbf{K}$ . These will be obtained directly from Einstein's equations (see *Chapter 5*).

## Chapter 3

# The 3 + 1 Equations for the Electromagnetic Field

After having introduced all the relevant geometric and kinematic quantities, we are now well-equipped to focus on the 3 + 1 formulation of specific relevant matter models in a 4-dimensional Lorentzian spacetime. Therefore, we consider a physical quantity, e.g. the electromagnetic field of a charged particle, given on the initial hypersurface<sup>1</sup>  $\varphi_0(\Sigma)$ , which is characterised by the spatial metric  $\mathbf{h}(t=0)$  and the extrinsic curvature  $\mathbf{K}(t=0)$ . Notice that we do not consider the “plasma” as the source of the gravitational field but rather something evolving in a specific given spacetime. A very illustrative example is the consideration of a rotating black hole surrounded by an accretion disk: the matter flow is the plasma which propagates on this Kerr background. Hence, the spatial metric is given by the Kerr metric and the extrinsic curvature depends on the chosen slicing [1].

Whilst the evolution of the gravitational field will be described and deduced in a later chapter, we are now interested in deriving the evolution equations of physical quantities, i.e. fields, on a curved background. We start with the electromagnetic field of a charge distribution and then move on to an ideal fluid (see *Chapter 4*) since their combination leads to *general relativistic ideal magnetohydrodynamics*, the most commonly used plasma model in Astrophysics.

### 3.1 Introduction

The easiest plasma model one might think of is simply a distribution of charged particles. It is well-known from the theory of electromagnetism that these electric charges generate an electric field  $\mathbf{E}$ . Further, when they

---

<sup>1</sup>We have chosen the initial time to be  $t = 0$

move they also produce a magnetic field  $\mathbf{B}$ . The origin of these fields as well as their interactions were unified by Maxwell in 1873, yielding the famous *Maxwell equations*:

$$\operatorname{div} \vec{E} = 4\pi\rho_e \quad (3.1)$$

$$\operatorname{rot} \vec{B} = \partial_t \vec{E} + 4\pi\vec{j} \quad (3.2)$$

$$\operatorname{div} \vec{B} = 0 \quad (3.3)$$

$$\operatorname{rot} \vec{E} = -\partial_t \vec{B} \quad (3.4)$$

Eq.(3.1) and Eq.(3.2) are known as the inhomogeneous Maxwell equations since they include the generators of the fields, which are the electric charge density  $\rho_e$  and the electric current  $\vec{j}$ . The other two equations are the homogeneous ones since no charges appear in them. Eq.(3.3) states the non-existence of magnetic monopoles. Furthermore, Eq.(3.2) is widely known as Ampère's law while Eq.(3.4) is the famous induction or Faraday law. The minus sign in Eq.(3.4) is a direct consequence of energy conservation respectively of Lenz's law.

In Relativity, the essential quantities are not the electric and magnetic field themselves, but an antisymmetric tensor of second rank, the *electromagnetic field tensor*  $\mathbf{F}$ , often also called the *Faraday tensor*. It is given by

$$\mathbf{F} = \mathbf{n} \wedge \mathbf{E} - *(\mathbf{B} \wedge \mathbf{n}), \quad (3.5)$$

where the  $*$ -operation denotes the *Hodge dual* [25]. In a chart chosen on  $M$  the above tensor equation is written as  $F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu - \epsilon_{\mu\nu\sigma\lambda} B^\sigma n^\lambda$ . Hence, we notice that  $\mathbf{F} = \mathbf{F}(x)$  is a tensor field on the spacetime manifold  $M$ . Due to the decomposition of  $\mathbf{F}$ , the electric and magnetic field are purely spatial and hence given by:

$$E_\mu = F_{\mu\nu} n^\nu \quad (3.6)$$

$$B^\mu = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} n_\nu F_{\alpha\beta}. \quad (3.7)$$

Due to the appearance of  $n^\mu$  in their definition, the physical quantities  $\mathbf{E}$  and  $\mathbf{B}$  are well-defined as sections of the pullback bundle of  $M$ , i.e.  $E = E(t, y^i)$  and  $B = B(t, y^i)$ .

Before turning to the covariant version of Maxwell's equations, we have to introduce another 4-vector representing the relativistic generalisation of the two inhomogeneities or field sources  $\rho_e$  and  $\vec{j}$ . In Relativity they are expressed by the 4-current  $J^\sigma$  defined as

$$J^\mu := \rho_e n^\mu + \varphi_{,i}^\mu j^i \quad (3.8)$$

Note that  $j^i$  is a purely spatial 3-vector representing the physical current and  $j^\mu$  is only its 4-dimensional extension. Again,  $J^\mu$  is well-defined as a section of the pullback bundle.

Now we are in the position to write down the governing equations of electromagnetism in a fully covariant form, namely:

$$\nabla_\mu F^{\mu\nu} = -4\pi J^\nu \quad (3.9)$$

$$\nabla_{[\sigma} F_{\mu\nu]} = 0. \quad (3.10)$$

Eq.(3.10) represents both inhomogeneous Maxwell equations whilst Eq.(3.11) is the covariant version of the homogeneous ones. After this brief summary of the electromagnetic interactions, we can now turn towards the evolution of the electric and the magnetic field on a given metric background but let us make one last remark.

*Remark:* The original Maxwell equations Eq.(3.1)-(3.4) represent a special case of the general 3 + 1 Maxwell equations which we derive in the next subsection. If we chose so-called *Gaussian coordinates*, namely  $N = 1$  and  $\beta = 0$ , and a hyperplane slicing ( $\mathbf{K}=0$ ) of Minkowski spacetime, the general equations turn out to take the standard form of Maxwell's equations. This, of course, corresponds to choosing  $\xi$  to be a time-translation Killing vector orthogonal to the hypersurface  $\varphi(\Sigma)$ .

## 3.2 Evolution of the Electric Field

In the case of electromagnetism there exist two different possibilities to derive the time evolution equations for the two fields: one could split Maxwell's equations Eq.(3.9) and Eq.(3.10) directly, but since we operate in a completely general setting<sup>2</sup> we have to be careful and must avoid any division by  $N$  as this operation is not defined. Therefore, we would have to apply a little "trick" and multiply Maxwell's equations by  $N$  before applying the projections as we will do in later chapters for reasons of simplicity<sup>3</sup>. But fortunately we are aware of the relevant quantities  $\mathbf{E}$  and  $\mathbf{B}$  and can therefore chose a different, completely general and far more elegant approach in order to determine the evolution equations. The starting point is the same as for the extrinsic curvature shown in *Appendix C*. Its mathematical basis is Eq.(2.66). Hence, we get (where the  $t$ -dependence of  $\varphi$  is suppressed):

$$\dot{E}_i(y) = \varphi_{,i}^\mu \mathcal{L}_\xi E_\mu(y) = \varphi_{,i}^\mu \mathcal{L}_\xi (F_{\mu\nu}(\varphi(y)) n^\nu(y)). \quad (3.11)$$

---

<sup>2</sup>Remember:  $N$  arbitrary.

<sup>3</sup>We have seen in Sec. 2.4.1 that the projection along the normal part of the time evolution vector is reasonable for slicings.



Using the definition of the Lie derivative, the explicit form of the time evolution vector  $\xi$  and Eq.(3.6) yields

$$\begin{aligned}
\dot{E}_i(y) &= \varphi_{,i}^\mu \mathcal{L}_\xi E_\mu(\varphi(y)) \\
&= \varphi_{,i}^\mu \{ \xi^\lambda \nabla_\lambda E_\mu + E_\lambda \nabla_\mu \xi^\lambda \} \\
&= \varphi_{,i}^\mu \{ \xi^\lambda \nabla_\lambda (F_{\mu\nu} n^\nu) + E_\lambda \nabla_\mu (N n^\lambda + \beta^\lambda) \} \\
&= \varphi_{,i}^\mu \{ \xi^\lambda (\nabla_\lambda F_{\mu\nu}) n^\nu + F_{\mu\nu} \xi^\lambda \nabla_\lambda n^\nu + \\
&\quad + N E_\lambda \nabla_\mu n^\lambda + n^\lambda E_\lambda \nabla_\mu N + E_\lambda \nabla_\mu \beta^\lambda \}. \tag{3.12}
\end{aligned}$$

Taking into account the definition of the inverse extended spatial metric  $h^{\alpha\beta} = g^{\mu\nu} + n^\mu n^\nu$  as well as Eq.(2.69) then gives

$$\begin{aligned}
\dot{E}_i &= \varphi_{,i}^\mu \{ N(h^{\lambda\nu} - g^{\lambda\nu}) \nabla_\lambda F_{\mu\nu} + F_{\mu\nu} \varphi_{,m}^\nu D^m N + \\
&\quad + \beta^\lambda \nabla_\lambda (F_{\mu\nu} n^\nu) - \beta^\lambda F_{\mu\nu} \nabla_\lambda n^\nu + \\
&\quad + \beta^\lambda F_{\mu\nu} \nabla_\lambda n^\nu - N E_\lambda K_\mu{}^\lambda + E_\lambda \nabla_\mu \beta^\lambda \} \\
&= \varphi_{,i}^\mu \{ N \varphi_{,m}^\lambda \varphi_{,n}^\nu h^{mn} \nabla_\lambda F_{\mu\nu} - N \nabla^\nu F_{\mu\nu} + F_{\mu\nu} \varphi_{,m}^\nu D^m N + \\
&\quad + \beta^\lambda \nabla_\lambda E_\mu - N E_\lambda K_\mu{}^\lambda + E_\lambda \nabla_\mu \beta^\lambda \} \tag{3.13}
\end{aligned}$$

*Remark:* We have written terms containing the extrinsic curvature since the right projector is applied to it. Otherwise, it would not be possible to identify  $\nabla_\alpha n_\beta = K_{\alpha\beta}$  since such an expression is not given explicitly without a projector in the context of a general setting. If we were using a foliation, we could make use of Eq.(2.26).

Making use of the fact that the pullback of  $\mathbf{F}$  yields a purely spatial antisymmetric rank 2 tensor governing the magnetic field defined as

$$\mathcal{B}_{ij} := \varphi_{,i}^\mu \varphi_{,j}^\nu F_{\mu\nu}, \tag{3.14}$$

gives the following important intermediate result

$$\begin{aligned}
\dot{E}_i &= \mathcal{L}_\beta E_i - 4\pi N \varphi_{,i}^\mu J_\mu - N E_k K_i{}^k + \\
&\quad + \mathcal{B}_{ik} D^k N + N \varphi_{,i}^\mu \varphi_{,m}^\lambda \varphi_{,n}^\nu h^{mn} \nabla_\lambda F_{\mu\nu}, \tag{3.15}
\end{aligned}$$

where we have also taken into account the definition of the Lie derivative and the fact that the pullback and the Lie derivative along the time evolution vector interchange in the case of a purely spatial object as well as the inhomogeneous Maxwell equations Eq.(3.9).

Although the above equation looks already quite fine, there is still a term (the very last one in Eq.(3.15)) we can not interpret a priori and thus it has

to be examined further.

$$\begin{aligned}
N\varphi_{,i}^\mu\varphi_{,m}^\lambda\varphi_{,n}^\nu h^{mn}\nabla_\lambda F_{\mu\nu} &= \\
&= N\varphi_{,i}^\mu\varphi_{,m}^\lambda\varphi_{,n}^\nu h^{mn}\{n_\mu\nabla_\lambda E_\nu + E_\nu\nabla_\lambda n_\mu - \\
&\quad - n_\nu\nabla_\lambda E_\mu - E_\mu\nabla_\lambda n_\nu + \nabla_\lambda\mathcal{B}_{\mu\nu}\} \\
&= N\varphi_{,i}^\mu\varphi_{,m}^\lambda\varphi_{,n}^\nu h^{mn}\{-E_\nu K_{\lambda\mu} + E_\mu K_{\nu\lambda} + \nabla_\lambda\mathcal{B}_{\mu\nu}\} \\
&= -NE_n K_i^n + N\mathcal{K}E_i + ND^n\mathcal{B}_{in},
\end{aligned} \tag{3.16}$$

where we have inserted the explicit construction of the Faraday tensor Eq.(3.5), the orthonormality condition and made use of the fact that  $\mathcal{B}$  is tangential to  $\Sigma$  in the last step. Altogether this gives the time evolution equation of the electric field  $\mathbf{E}$ :

$$\dot{E}_i - \mathcal{L}_\beta E_i = D^k(N\mathcal{B}_{ik}) - 4\pi Nj_i - 2NE_k K_i^k + N\mathcal{K}E_i. \tag{3.17}$$

Obviously, the above equation does not look like the famous Ampère law (see Eq.(3.2)) since we have not used the physical magnetic field itself but this rather practical rank 2 tensor  $\mathcal{B}$ . Therefore, a transformation is needed. Eq.(3.5) reveals immediately the relation between the “magnetic field tensor” and the magnetic field itself:

$$\mathcal{B}_{ij} = \epsilon_{ijk}B^k, \tag{3.18}$$

where  $\epsilon_{ijk}$  is the metric volume form given in local coordinates. Let us recall the definition of the metric volume form :  $\epsilon := e^1 \wedge \dots \wedge e^n$ . Hence, in local coordinates this writes as  $\epsilon_{ijk} = \sqrt{h}\epsilon(i, j, k)$ , where  $\epsilon(i, j, k)$  are the components of the volume form given in an orthonormal basis and therefore represent the well-known permutation tensor or Levi-Civita symbol, which takes the value +1 for an even permutation of  $(i, j, k)$ , -1 for an odd permutation and 0 if  $i = j$  or  $j = k$  or  $k = i$ .  $h$  is the determinant of the spatial metric.

Inserting Eq.(3.18) into the non-standard evolution equation Eq.(3.17) yields

$$\begin{aligned}
D^k(N\mathcal{B}_{ik}) &= D^k(NB^n\epsilon_{ikn}) = \epsilon_{ikn}D^k(NB^n) \\
&= (D \times NB)_i.
\end{aligned} \tag{3.19}$$

We have taken into account that the covariant derivative of the metric volume form vanishes, since it is an isometry of  $\mathbf{h}$  mathematically spoken. Therefore it conserves volumes. Further we have used the standard definition of the cross product in three dimensions:  $(rotX)_i = (D \times X)_i = \epsilon_{ijk}D^jX^k$ . Now, with this transformation the well-know Ampère law in its 3 + 1 form can be written down:

$$(\partial_t - \mathcal{L}_\beta)E_i = (D \times NB)_i - 4\pi Nj_i - 2NE_k K_i^k + N\mathcal{K}E_i. \tag{3.20}$$

Note that the above equation only represents the evolution of the  $i$ -th component of the electric field. Since the field is a 3-vector, we actually have three equations, one for each component.

### 3.3 Evolution of the Magnetic Field

After having treated the electric field, we can now focus on the magnetic field  $\mathbf{B}$  and derive its time evolution when given on a metric background. As already mentioned in the subsection before, one could also project Maxwell's equations directly, but we stick to the mathematically more elegant formalism. Using Eq.(2.66) it holds that

$$\dot{\mathcal{B}}_{ij} = \varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\xi F_{\mu\nu}. \quad (3.21)$$

It is very convenient to derive the evolution equation using the magnetic field tensor. Once again, we perform the transformation into the physical quantity  $\mathbf{B}$  at the end. Now, using the definition of the Lie derivative for a second rank co-tensor field gives:

$$\begin{aligned} \dot{\mathcal{B}}_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu \{ \xi^\lambda \nabla_\lambda F_{\mu\nu} + F_{\mu\lambda} \nabla_\nu \xi^\lambda + F_{\lambda\nu} \nabla_\mu \xi^\lambda \} \\ &= \varphi_{,i}^\mu \varphi_{,j}^\nu \{ n_\mu \xi^\lambda \nabla_\lambda E_\nu + E_\nu \xi^\lambda \nabla_\lambda n_\mu - n_\nu \xi^\lambda \nabla_\lambda E_\mu - \\ &\quad - E_\mu \xi^\lambda \nabla_\lambda n_\nu + \xi^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} + N F_{\mu\lambda} \nabla_\nu n^\lambda + \\ &\quad + n^\lambda F_{\mu\lambda} \nabla_\nu N + F_{\mu\lambda} \nabla_\nu \beta^\lambda + N F_{\lambda\nu} \nabla_\mu n^\lambda + \\ &\quad + n^\lambda F_{\lambda\nu} \nabla_\mu N + F_{\lambda\nu} \nabla_\mu \beta^\lambda \}, \end{aligned} \quad (3.22)$$

where we also used the decomposition of the Faraday tensor Eq.(3.5) as well as the explicit form of the evolution vector. Notice, that all terms are well-defined since the right projection operators are applied. Taking the orthonormality condition into account yields immediately

$$\begin{aligned} \dot{\mathcal{B}}_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu \{ -E_\nu \xi^\lambda K_{\lambda\mu} + E_\mu \xi^\lambda K_{\lambda\nu} + \\ &\quad + N n^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} + \beta^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} - \\ &\quad - N F_{\mu\lambda} K_\nu^\lambda + E_\mu \nabla_\nu N + \\ &\quad + n_\mu E_\lambda \nabla_\nu \beta^\lambda - n_\lambda E_\mu \nabla_\nu \beta^\lambda + \mathcal{B}_{\mu\lambda} \nabla_\nu \beta^\lambda - \\ &\quad - N F_{\lambda\nu} K_\mu^\lambda - E_\nu \nabla_\mu N + \\ &\quad + n_\lambda E_\nu \nabla_\mu \beta^\lambda - n_\nu E_\lambda \nabla_\mu \beta^\lambda + \mathcal{B}_{\lambda\nu} \nabla_\mu \beta^\lambda \}. \end{aligned} \quad (3.23)$$

Further, applying that the contraction  $K_{\lambda\mu}n^\lambda$  vanishes and again putting in the orthonormality condition gives

$$\begin{aligned}\dot{\mathcal{B}}_{ij} = & \varphi_{,i}^\mu \varphi_{,j}^\nu \{ -E_\nu \beta^\lambda K_{\lambda\mu} + E_\mu \beta^\lambda K_{\lambda\nu} + \\ & + N n^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} + \beta^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} - N \mathcal{B}_{\mu\lambda} K_\nu^\lambda + \\ & + E_\mu \nabla_\nu N + E_\mu \beta^\lambda \nabla_\nu n_\lambda + \mathcal{B}_{\mu\lambda} \nabla_\nu \beta^\lambda - \\ & - N \mathcal{B}_{\lambda\nu} K_\mu^\lambda - E_\nu \nabla_\mu N - E_\nu \beta^\lambda \nabla_\mu n_\lambda + \mathcal{B}_{\lambda\nu} \nabla_\mu \beta^\lambda \}. \end{aligned} \quad (3.24)$$

This intermediate result still looks a bit bulky but fortunately we can make use of Eq.(C.8). After inserting this equation into Eq.(3.24) nearly all terms cancel and therefore we get the following important intermediate result:

$$\begin{aligned}\dot{\mathcal{B}}_{ij} = & \varphi_{,i}^\mu \varphi_{,j}^\nu \{ \mathcal{L}_\beta \mathcal{B}_{\mu\nu} + N n^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} - N \mathcal{B}_{\mu\lambda} K_\nu^\lambda - \\ & - N \mathcal{B}_{\lambda\nu} K_\mu^\lambda + E_\mu \nabla_\nu N - E_\nu \nabla_\mu N \}. \end{aligned} \quad (3.25)$$

As already aforementioned interchanges the pullback and the Lie derivative of an object tangential to  $\Sigma$  along a purely spatial vector field, which concretely means in the case of the magnetic field tensor

$$\varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\beta \mathcal{B}_{\mu\nu} = \mathcal{L}_\beta \mathcal{B}_{ij}. \quad (3.26)$$

Before we make use of this fact, we have to specify the second term on the right-hand side of Eq.(3.25), since we cannot interpret it a priori. Expressing  $\mathcal{B}$  in terms of the electromagnetic field tensor (see Eq.(3.5)) and making use of Eq.(2.71) leads to

$$\begin{aligned}\varphi_{,i}^\mu \varphi_{,j}^\nu N n^\lambda \nabla_\lambda \mathcal{B}_{\mu\nu} = & \varphi_{,i}^\mu \varphi_{,j}^\nu N n^\lambda \nabla_\lambda \{ F_{\mu\nu} - n_\mu E_\nu + n_\nu E_\mu \} \\ = & \varphi_{,i}^\mu \varphi_{,j}^\nu \{ N n^\lambda \nabla_\lambda F_{\mu\nu} + E_\mu \gamma_\nu^k D_k N - E_\nu \gamma_\mu^k D_k N \} \\ = & E_i D_j N - E_j D_i N + \varphi_{,i}^\mu \varphi_{,j}^\nu N n^\lambda \nabla_\lambda F_{\mu\nu}, \end{aligned} \quad (3.27)$$

where we have again used the orthonormality condition. As a look at the last term in the just derived equation suggests, we are not finished yet. Exactly this term has to be rehandled since no a priori conclusions can be made. Making use of the homogeneous Maxwell equations gives

$$\begin{aligned}\varphi_{,i}^\mu \varphi_{,j}^\nu N n^\lambda \nabla_\lambda F_{\mu\nu} = & \varphi_{,i}^\mu \varphi_{,j}^\nu N n^\lambda \{ -\nabla_\nu F_{\lambda\mu} - \nabla_\mu F_{\nu\lambda} \} \\ = & \varphi_{,i}^\mu \varphi_{,j}^\nu N \{ \nabla_\nu (F_{\lambda\mu} n^\lambda) + F_{\lambda\mu} \nabla_\nu n^\lambda - \nabla_\mu (F_{\nu\lambda} n^\lambda) + F_{\nu\lambda} \nabla_\mu n^\lambda \} \\ = & \varphi_{,i}^\mu \varphi_{,j}^\nu N \{ \nabla_\nu E_\mu - \nabla_\mu E_\nu - F_{\lambda\mu} K_\nu^\lambda - F_{\nu\lambda} K_\mu^\lambda \} \\ = & N D_j E_i - N D_i E_j - \varphi_{,i}^\mu \varphi_{,j}^\nu N \{ \mathcal{B}_{\lambda\mu} K_\nu^\lambda + \mathcal{B}_{\nu\lambda} K_\mu^\lambda \}. \end{aligned} \quad (3.28)$$

Taking the last two auxiliary results and inserting them into the original equation Eq.(3.25) provides the final result in terms of  $\mathcal{B}$ :

$$\dot{\mathcal{B}}_{ij} - \mathcal{L}_\beta \mathcal{B}_{ij} = D_j (N E_i) - D_i (N E_j) \quad (3.29)$$

In order to reformulate the  $3 + 1$  form of the induction law in terms of the physical quantity  $\mathbf{B}$  we once again have to apply the transformation to it. Be aware that the transformation here is a bit more cumbersome than it was in the case of the electric field as it includes two more complex terms. Let us start with the time derivative term: since the partial derivative obeys the Leibniz rule we get immediately

$$\begin{aligned}\dot{\mathcal{B}}_{ij} &= \partial_t(\epsilon_{ijk}B^k) = \partial_t(\sqrt{h}\epsilon(i, j, k)B^k) = \\ &= \frac{1}{2} \frac{\dot{h}}{\sqrt{h}} \epsilon(i, j, k)B^k + \sqrt{h}\epsilon(i, j, k)\dot{B}^k.\end{aligned}\quad (3.30)$$

It is well-known that the derivative of the determinant is explicitly given by the following relation, which can be found in most books on *Analysis and Calculus*:

$$\dot{h} = h(h^{ij}\dot{h}_{ij}) \quad (3.31)$$

Since the time derivative of the spatial metric has already been deduced in *Chapter 2* Eq.(2.77) it follows that

$$\dot{h} = h(-2NK_{ij}h^{ij} + h^{ij}\mathcal{L}_\beta h_{ij}) = -2N\mathcal{K}h + hh^{ij}\mathcal{L}_\beta h_{ij}. \quad (3.32)$$

Making use of the definition of the Lie derivative turns the last term on the right-hand side of the above equation into

$$\begin{aligned}h^{ij}\mathcal{L}_\beta h_{ij} &= h^{ij}(\beta^k D_k h_{ij} + h_{kj} D_i \beta^k + h_{ik} D_j \beta^k) \\ &= \delta_k^i D_i \beta^k + \delta_k^j D_j \beta^k \\ &= 2D_k \beta^k.\end{aligned}\quad (3.33)$$

Altogether the time derivative of the determinant of the spatial metric is therefore given by

$$\dot{h} = h(-2N\mathcal{K} + 2D_k \beta^k). \quad (3.34)$$

Applying the above mentioned result to the initial equation Eq.(3.30) determines the  $\mathbf{B}$ -field expression for the term containing the time derivative:

$$\begin{aligned}\dot{\mathcal{B}}_{ij} &= \sqrt{h}(-N\mathcal{K} + D_m \beta^m)\epsilon(i, j, k)B^k + \sqrt{h}\epsilon(i, j, k)\dot{B}^k \\ &= -\epsilon_{ijk}N\mathcal{K}B^k + \epsilon_{ijk}(D_m \beta^m)B^k + \epsilon_{ijk}\dot{B}^k.\end{aligned}\quad (3.35)$$

Secondly, we have to work out the other term on the left-hand side of Eq.(3.29). The definition of  $\mathcal{B}$  and the Lie derivative yield

$$\mathcal{L}_\beta \mathcal{B}_{ij} = \mathcal{L}_\beta(\epsilon_{ijk}B^k) = B^k \mathcal{L}_\beta \epsilon_{ijk} + \epsilon_{ijk} \mathcal{L}_\beta B^k \quad (3.36)$$

$$B^k \mathcal{L}_\beta \epsilon_{ijk} = \epsilon_{mjk} D_i \beta^m + \epsilon_{imk} D_j \beta^m + \epsilon_{ijm} D_k \beta^m \quad (3.37)$$

Consequently, the left-hand side of the evolution equation is then

$$\begin{aligned}\dot{\mathcal{B}}_{ij} - \mathcal{L}_\beta \mathcal{B}_{ij} = & -\epsilon_{ijk} N \mathcal{K} B^k + \epsilon_{ijk} (D_m \beta^m) B^k + \epsilon_{ijk} \dot{B}^k - \\ & - \epsilon_{ijk} \mathcal{L}_\beta B^k - B^k \epsilon_{mjk} D_i \beta^m - \\ & - B^k \epsilon_{imk} D_j \beta^m - B^k \epsilon_{ijm} D_k \beta^m.\end{aligned}\quad (3.38)$$

Though we got rid of the terms containing the spatial part of the Faraday tensor, the resulting expression in terms of the real magnetic field does not look too handy but contracting it with the inverse volume form gives

$$\begin{aligned}\epsilon^{ijn} (\dot{\mathcal{B}}_{ij} - \mathcal{L}_\beta \mathcal{B}_{ij}) = & 2\delta_k^n (-N \mathcal{K} B^k + B^k D_m \beta^m + \dot{B}^k - \mathcal{L}_\beta B^k) - \\ & - (\delta_m^i \delta_k^n - \delta_k^i \delta_m^n) B^k D_i \beta^m - \\ & - (\delta_m^j \delta_k^n - \delta_k^j \delta_m^n) B^k D_j \beta^m - 2B^k D_k \beta^n,\end{aligned}\quad (3.39)$$

where we have used two important, well-known identities for  $\epsilon$ , namely

$$\epsilon_{imn} \epsilon^{jmn} = 2\delta_i^j \quad (3.40)$$

$$\epsilon_{ijk} \epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m. \quad (3.41)$$

As expected we get eventually

$$\epsilon^{ijn} (\dot{\mathcal{B}}_{ij} - \mathcal{L}_\beta \mathcal{B}_{ij}) = 2\dot{B}^n - 2\mathcal{L}_\beta B^n - 2N \mathcal{K} B^n. \quad (3.42)$$

Until now we have only “massaged” the left-hand side of Eq.(3.29). Since its right-hand side does not vanish, the inverse volume form has to be applied to it as well, giving

$$\epsilon^{ijn} (D_j (N E_i) - D_i (N E_j)) = -2(D \times N E)^n, \quad (3.43)$$

where the standard expression for the cross product in terms of the  $\epsilon$ -symbol was used. All in all we obtain the final evolution equation:

$$\dot{B}^i - \mathcal{L}_\beta B^i = N \mathcal{K} B^i - (D \times N E)^i. \quad (3.44)$$

This clearly looks like the famous induction law, but a comparison with the evolution equation for the electric field Eq.(3.20) shows an obvious difference: the above equation does not contain a term of the type  $K_{ik} B^k$ . The reason for this is that we have developed the time evolution for the electric co-vector field whereas Eq.(3.44) is the evolution equation for the magnetic vector field. For consistency we transform these equations into a relation in terms of the magnetic co-vector field. This is done by inserting  $B^i = h^{ij} B_j$  yielding

$$\mathcal{L}_\beta B^i = h^{ij} \mathcal{L}_\beta B_j + 2B_j N K^{ij} + B_j \dot{h}^{ij} \quad (3.45)$$

$$\dot{B}^i = \partial_t (h^{ij} B_j) = B_j \dot{h}^{ij} + h^{ij} \dot{B}_j, \quad (3.46)$$

where again the definition of the Lie derivative was taken into account. The application of the metric  $\mathbf{h}$  gives the ultimate result:

$$(\partial_t - \mathcal{L}_\beta)B_i = N\mathcal{K}B_i + 2NK_i^k B_k - (D \times NE)_i. \quad (3.47)$$

This is the famous Faraday law in its 3+1 version referred to as the evolution equation of the magnetic field  $\mathbf{B}$ .

### 3.4 Projected Divergence Equations

By now we have only found six equations governing the dynamics of the electric and the magnetic field. But Maxwell's equations are a total of eight equations. Therefore, two more equations, namely the divergence equations, are still missing and have to be found in order to close the Maxwell system.

In order to obtain the scalar equations describing the sources of the electric field, the inhomogeneous Maxwell equations have to be contracted with the unit normal. Again, like in Sec. 2.3.6, the deduced formulae do not depend on the extension of  $\mathbf{n}$  off a hypersurface  $\varphi_t(\Sigma)$ . Hence, we get

$$n_\nu \nabla_\mu F^{\mu\nu} = -E^\nu n^\mu \nabla_\mu n_\nu + \nabla_\mu E^\mu \quad (3.48)$$

Since the electric field  $\mathbf{E}$  is tangential to  $\Sigma$ , the right-hand side exactly prescribes the covariant divergence of the electric field  $\mathbf{E}$  with respect to the spatial metric  $\mathbf{h}$ :

$$\begin{aligned} D_i E^i &= \varphi_{,i}^\mu \gamma_\nu^i \nabla_\mu E^\nu \\ &= h^\mu{}_\nu \nabla_\mu E^\nu \\ &= (\delta^\mu{}_\nu + n^\mu n_\nu) \nabla_\mu E^\nu \\ &= \nabla_\mu E^\mu - E^\nu n^\mu \nabla_\mu n_\nu \\ &= \nabla_\mu E^\mu - E^i D_i N. \end{aligned} \quad (3.49)$$

On the other hand, the contraction of the right-hand side of the inhomogeneous Maxwell equations yields  $4\pi\rho_e$ . Together, this gives the projected divergence equation for the electric field:

$$D_i E^i = 4\pi\rho_e. \quad (3.50)$$

For retrieving the second scalar Maxwell equation covering the fact, that no magnetic monopoles exist, we rewrite Eq.(3.10) in a slightly different but absolutely equivalent form, namely

$$\epsilon^{\mu\nu\sigma\lambda} \nabla_\sigma F_{\mu\nu} = 0. \quad (3.51)$$

In the previous section we have already used two important identities for the 3-dimensional  $\epsilon$ -tensor. The 4-dimensional version of Eq.(3.42) then writes as

$$\epsilon^{\mu\nu\sigma\lambda}\epsilon_{\mu\nu\alpha\beta} = -2(\delta_\alpha^\sigma\delta_\beta^\lambda - \delta_\alpha^\lambda\delta_\beta^\sigma). \quad (3.52)$$

A comprehensive treatment of the volume form and important relations can for example be found in [41].

Inserting the decomposition of the electromagnetic field tensor and using the identity Eq.(3.52) then gives

$$\begin{aligned} \epsilon^{\mu\nu\sigma\lambda}\nabla_\sigma F_{\mu\nu} &= 2\epsilon^{\mu\nu\sigma\lambda}(n_\mu\nabla_\sigma E_\nu - E_\mu\nabla_\sigma n_\nu) + \\ &\quad + 2(\delta_\alpha^\sigma\delta_\beta^\lambda - \delta_\alpha^\lambda\delta_\beta^\sigma)n^\beta\nabla_\sigma B^\alpha - \\ &\quad - 2(\delta_\alpha^\sigma\delta_\beta^\lambda - \delta_\alpha^\lambda\delta_\beta^\sigma)B^\alpha\nabla_\sigma n^\beta \\ &= 0. \end{aligned} \quad (3.53)$$

Hence, we get

$$\begin{aligned} 0 &= \epsilon^{\mu\nu\sigma\lambda}(n_\mu\nabla_\sigma E_\nu - E_\mu\nabla_\sigma n_\nu) + \\ &\quad + n^\lambda\nabla_\alpha B^\alpha - n^\beta\nabla_\beta B^\lambda - B^\sigma\nabla_\sigma n^\lambda - B^\lambda\mathcal{K}. \end{aligned} \quad (3.54)$$

In order to obtain the divergence, the above intermediate result has to be contracted with  $\mathbf{n}^4$ , leading to

$$\begin{aligned} 0 &= -n_\lambda E_\mu\nabla_\sigma(\epsilon^{\mu\nu\sigma\lambda}n_\nu) + n_\lambda n_\nu E_\mu\nabla_\sigma(\epsilon^{\mu\nu\sigma\lambda}) - \\ &\quad - \nabla_\lambda B^\lambda - n_\lambda n^\beta\nabla_\beta B^\lambda, \end{aligned} \quad (3.55)$$

where we have made use of the fact that the 4-dimensional  $\epsilon$ -tensor is totally antisymmetric in all indices and therefore any contraction over two indices with a symmetric object, e.g.  $n \otimes n$ , vanishes. Furthermore, we have used that the scalar product of two orthogonal vectors is zero. Additionally, using the relation between the covariant divergence with respect to  $\mathbf{g}$  and the one with respect to  $\mathbf{h}$  (see Eq.(3.49)) and the fact that the covariant derivative of the 3- and 4-volume form vanishes, finally yields the well-known equation describing the non-existence of magnetic charges in its 3 + 1 form:

$$\nabla_\lambda B^\lambda - B^\lambda n^\alpha \nabla_\alpha n_\lambda \equiv D_i B^i = 0. \quad (3.56)$$

### 3.5 The 3+1 Maxwell system

Combining the results established in the previous sections, we find that the initial value formulation of the Maxwell system consists of the six evolution

---

<sup>4</sup>Again, the derived results are independent of the extension of  $\mathbf{n}$  off the hypersurface.



equations for the electric and the magnetic field, as well as of two constraint equations, which coincide with the equations found by [2] in the case of a regular foliation:

*The evolution equations*<sup>5</sup>:

$$(\partial_t - \mathcal{L}_\beta)E_i = (D \times NB)_i - 4\pi N j_i - 2NE_k K_i^k + N\mathcal{K}E_i \quad (3.57)$$

$$(\partial_t - \mathcal{L}_\beta)B_i = N\mathcal{K}B_i + 2NK_i^k B_k - (D \times NE)_i \quad (3.58)$$

*The constraint equations*:

$$D_i E^i = 4\pi \rho_e \quad (3.59)$$

$$D_i B^i = 0 \quad (3.60)$$

**Lemma 1** *The 3 + 1 Maxwell system Eq.(3.57) and Eq.(3.58) as well as Eq.(3.59) and Eq.(3.60) is equivalent to the original Maxwell equations given by Eq.(3.9) and Eq.(3.10) and represents its well-posed initial value formulation.*

**Proof:** *We do not present the proof here since there exists an extensive amount of literature on this. The equivalence of the original and the 3 + 1 version of Maxwell's equations is more or less apparent from the explicit calculations performed here. In order to proof the well-posedness we refer to literature, see for example [42] or [37]. The common proof makes use of potentials and the Lorenz gauge, but well-posedness can also be shown by using the above equations directly.*

□

---

<sup>5</sup>The vacuum evolution equations can be found in [6].

## Chapter 4

# The 3+1 Formulation of General Relativistic Hydrodynamics

### 4.1 Introduction

In the previous chapter we have discussed the case of a purely electromagnetic plasma, i.e. moving charges, on a given background. However, most astrophysical configurations involve matter sources like stars or accretion discs and therefore it is not adequate to use a purely electromagnetic plasma. Most of the mentioned systems involve *gases* and therefore it is natural to use the well-established theory of *fluid dynamics* to model them which is a standard procedure in Astrophysics.

Fluid dynamics describes matter as a continuum: the volume or fluid element considered is infinitesimally small compared to the entire system, but it still contains a large number of particles. It is obvious, that the fluid description is only an approximation. In order to be an appropriate one, the distance between the particles and the mean free path between collisions must be small compared to the physical length of the system.

Usually, the state of the entire fluid is characterised by five quantities: the three components of the velocity field  $\mathbf{v}$  of a volume element and two thermodynamical quantities, for instance the mass-energy density  $\epsilon$  and the pressure  $p$ . The thermodynamical quantities are related via an *equation of state* (EOS), i.e.

$$p = f(\epsilon, T), \tag{4.1}$$

where  $T$  is the temperature. It is clear that the choice of the EOS depends on the specific fluid model. For a brief discussion of the thermodynamics of a perfect fluid see below.

In the context of modelling astrophysically relevant matter, a specific type of a fluid is commonly used as a first approximation, namely a *perfect*

or **ideal fluid**. Such a fluid describes freely moving particles, which interact via collisions and can therefore exchange energy and momentum. Further, no particles are created or destroyed. Hence, the number of particles per unit volume stays constant. On the other hand side, more general but also more complex fluid models can be considered, so-called imperfect fluids. Normally, these models also include non-linearities like heat conduction and viscosity. Hence, we call an ideal fluid *inviscid*. Furthermore, the fluid can be chosen to consist of several components like ions and free electrons for example [14].

In order to actually describe and follow the *motion of the fluid*, two distinct approaches are known: the first possibility, the so-called *Lagrangian approach*, ties a chosen coordinate system to an individual fluid element and flows with it in time. In the second approach a coordinate system fixed in spacetime is considered and the motion of the fluid is described as seen from an observer at rest in this specific frame. This approach is known as the *Eulerian* one. Mathematically, the main difference is encoded in the so-called *convective derivative*. However, it is common to follow the Eulerian approach, since in the case of an imperfect fluid shear forces can easily distort the coordinate system tied to a volume element [1].

The non-relativistic dynamical equations governing the fluid motion of a perfect fluid are the *Euler equations*

$$\partial_t v^i + (v^j D_j) v^i = -\frac{1}{\epsilon} D_i p. \quad (4.2)$$

In the case of a viscid fluid the analogous equations are referred to as the *Navier-Stokes equations*. In the following we will only consider ideal fluids and derive the equations of motion in the  $3+1$  form. Therefore, we are again neglecting the gravitational field generated by the fluid itself whose motion shall be determined.

The description of general relativistic hydrodynamics here will be very short, since we focus on the deduction of the  $3+1$  evolution equations of the physical quantities. Naturally, there exists a large number of books dedicated to the theory of fluid dynamics. At this point we strongly recommend to the book by Landau and Lifšic [27] for a comprehensive treatment of the subject.

## 4.2 Fluid kinematics and thermodynamics

The main goal of fluid dynamics is to predict the motion of a fluid from given initial conditions. In non-relativistic hydrodynamics the main quantity is the velocity field  $\mathbf{v}$  of the fluid. Given appropriate initial data, the equation for the velocity field can then be solved and the entire motion of the fluid predicted.

Let us mention again, that we neglect the gravitational field of the fluid. Therefore, we do not consider the Einstein field equations and assume  $\mathbf{g}$  to be the metric of a given background. The fluid is then called a *test fluid*. In relativistic hydrodynamics the essential kinematic quantity to study is again the velocity field though one has to deal with a 4-velocity field  $u$  instead, which is a vector field on the spacetime manifold  $M$ . The integral curves of this vector field are the worldlines of the fluid elements. In fluid dynamics these trajectories are known as the *streamlines* of the fluid particles. Due to causality, the velocity field is assumed to be timelike. By definition it then satisfies the *normalisation condition*

$$u^\mu u_\mu = -1. \quad (4.3)$$

Since we are now aware of the main kinematic variable, namely the 4-velocity  $\mathbf{u}$ , we turn towards the thermodynamical description of a perfect fluid. As aforementioned, besides the three kinematic functions, there are two thermodynamical variables, such as the *energy density* and the *particle number density*, needed, in order to determine the motion of the fluid. These variables characterise the thermodynamical state of fluid. They are measured in the momentarily co-moving frame of reference, i.e. the local rest frame of the fluid and therefore appointed “proper”. The various variables are:

$\bar{\epsilon}$	proper total energy density
$\bar{n}$	proper particle number density
$\bar{p}$	principal pressure
T	temperature
s	specific entropy
e	specific internal energy
v	specific volume
h	specific enthalpy

Some of these variables depend on each other like

$$v = \frac{1}{\bar{n}}, \quad \epsilon = \bar{n}(1 + e), \quad h = e + \bar{p}v. \quad (4.4)$$

It is important to note that we are not interested in a microscopic description of the fluid. The above introduced thermodynamical variables are macroscopic quantities since they characterise an entire ensemble of particles. Hence, they can be interpreted as the averages of microscopic quantities taken over a suitable volume. As we operate on the level of this macroscopic description of the fluid, the *first law of thermodynamics* is suitable:

$$de = Tds - \bar{p}dv. \quad (4.5)$$

It states that the internal energy of the system can be changed due to additional thermal energy or mechanical work. Commonly, the equation of state

is given in the form of Eq.(4.1). Together with the first law of thermodynamics and the relations between the various thermodynamical variables, one can then calculate all variables and thus obtain the full thermodynamical description of the fluid when the EOS is given.

As mentioned in the introduction part, we do not consider any particle creation or annihilation processes. This implies the *conservation of the number of particles*:

$$\nabla_\mu(\bar{n}u^\mu) = 0. \quad (4.6)$$

In the non-relativistic limit the particle number conservation corresponds to the *continuity equation*

$$\partial_t \bar{\epsilon} + D_i(\bar{\epsilon} v^i) = 0 \quad (4.7)$$

In General Relativity matter is characterised by its corresponding stress-energy tensor  $\mathbf{T}$  (see *Appendix D* for further information). The stress-energy tensor describing an ideal fluid is then given by

$$T_{\mu\nu} = (\bar{\epsilon} + \bar{p})u_\mu u_\nu + \bar{p}g_{\mu\nu}. \quad (4.8)$$

Since we consider an ideal fluid, we further have local energy and momentum conservation. Hence, all together the governing covariant conservation laws take the form

$$\nabla_\mu(\bar{n}u^\mu) = 0 \quad (4.9)$$

$$\nabla^\mu T_{\mu\nu} = 0. \quad (4.10)$$

The above system of equations therefore is the relativistic generalisation of the Euler system Eq.(4.2) and Eq.(4.7). Together with an EOS the motion of a relativistic ideal fluid can be determined completely.

### 4.3 The 3+1 Euler equations

According to the explanation in the previous subsection, the main quantity is the 4-velocity of the fluid. Any vector field on the spacetime manifold  $M$  can be decomposed into its parts normal and tangential to the initial hypersurface as we have seen in Sec. 2.2. Thus, the 4-velocity is explicitly given by

$$u^\mu = \gamma(n^\mu + \varphi^\mu_{,k} v^k), \quad (4.11)$$

where  $\gamma = (1 - v^i v_i)^{-\frac{1}{2}}$  is the *Lorentz factor* and  $\mathbf{v}$  the 3-velocity of the fluid, the main kinematic quantity in the non-relativistic Euler equations. Of course, it is also the essential variable in the 3+1 form and therefore the relevant dynamical quantity.

As mentioned before, we are interested in deriving the evolution equations using the Eulerian approach. Therefore, we first have to define the energy density, the momentum density, the particle number density and the stress tensor with respect to the normal  $\mathbf{n}$  and not to the 4-velocity field  $\mathbf{u}$ . Hence, as given by Eq.(D.1), the energy density w.r.t. the normal is then

$$\begin{aligned}\epsilon &= T_{\mu\nu}n^\mu n^\nu \\ &= (\bar{\epsilon} + \bar{p})u_\mu n^\mu u_\nu n^\nu + \bar{p}g_{\mu\nu}n^\mu n^\nu \\ &= \gamma^2(\bar{\epsilon} + \bar{p}) - \bar{p},\end{aligned}\tag{4.12}$$

where we have made use of Eq.(4.11) as well as of the normalisation and orthonormality condition for  $\mathbf{n}$ .

The particle number density w.r.t. the normal can be obtained via

$$\begin{aligned}n &= -\bar{n}u^\mu n_\mu \\ &= \gamma\bar{n}.\end{aligned}\tag{4.13}$$

Further, the purely spatial part of the stress-energy tensor, the so-called *stress-tensor*, is determined by

$$\begin{aligned}S_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu T_{\mu\nu} \\ &= \varphi_{,i}^\mu \varphi_{,j}^\nu \{(\bar{\epsilon} + \bar{p})u_\mu u_\nu + \bar{p}g_{\mu\nu}\} \\ &= \gamma^2(\bar{\epsilon} + \bar{p})v_i v_j + \bar{p}h_{ij} \\ &= (\epsilon + \bar{p})v_i v_j + \bar{p}h_{ij},\end{aligned}\tag{4.14}$$

where we have used Eq.(4.12).

Eventually, the momentum density is given by

$$\begin{aligned}p_i &= -\varphi_{,i}^\nu n^\mu T_{\mu\nu} \\ &= \gamma^2(\bar{\epsilon} + \bar{p})v_i \\ &= (\epsilon + \bar{p})v_i.\end{aligned}\tag{4.15}$$

*Remark:* Generally, the so-called *principal pressures* are defined as the eigenvalues  $p_1, p_2, p_3$  of the stress tensor  $\mathbf{S}$ . They coincide in the case of an ideal fluid and are simply referred as the pressure  $\bar{p}$ . [10]

Finally, we have collected all essential relations and definitions to start deriving the evolution equations of the velocity  $\mathbf{v}$ . As already mentioned before there are two ways to deduce the evolution equations: either to directly use the relation between the time derivative and the Lie derivative, namely

$$\dot{v}_i = \varphi_{,i}^\mu \mathcal{L}_\xi(u_\mu - n_\lambda u^\lambda u_\mu),\tag{4.16}$$

or the indirect way of projecting Eq.(4.9) onto the hypersurface and reconstructing the time derivative via the evolution vector  $\xi$ . In order to do so, one has to multiply the conservation equation with the lapse function  $N$ . This looks like a “trick” and is certainly less elegant but using Eq.(4.16) directly turns out to lead to quite complex terms and relations, whereas the projective way is straightforward. Therefore, we perform the derivation of the evolution equations using this time the mathematically less elegant way contrary to the derivation of the evolution equations for the electric and magnetic field, where we have used the other way.

Hence, our starting point is Eq.(4.11), where we insert the stress-energy tensor in its general decomposition given by Eq.(D.7). It follows immediately

$$\begin{aligned}
N\nabla^\mu T_{\mu\nu} &= Ng^{\mu\lambda}\nabla_\lambda T_{\mu\nu} \\
&= Ng^{\mu\lambda}\nabla_\lambda \{S_{\mu\nu} + n_\mu p_\nu + p_\mu n_\nu + \epsilon n_\mu n_\nu\} \\
&= N\nabla_\lambda S^\lambda{}_\nu - N\mathcal{K}p_\nu + Nn^\lambda\nabla_\lambda p_\nu + Nn_\nu\nabla_\lambda p^\lambda + \\
&\quad + Np^\lambda\nabla_\lambda n_\nu - N\epsilon\mathcal{K}n_\nu + \epsilon Nn^\lambda\nabla_\lambda n_\nu + n_\nu Nn^\lambda\nabla_\lambda \epsilon \\
&= 0,
\end{aligned} \tag{4.17}$$

where we have taken into account equation Eq.(2.23) as well as the facts, that the covariant derivative of the metric and the scalar product between the unit normal and any extended spatial vector vanish.

*Remark:* Let us point out again that the multiplication with the lapse function  $N$  is necessary to obtain well-defined expressions when considering a general slicing (see Sec. 2.4.1).

In order to get the time evolution equation of the total energy density, which corresponds to Eq.(4.7) in the non-relativistic limit, the contraction of Eq.(4.17) is needed. The contraction yields

$$\begin{aligned}
0 &= n^\nu \{ N\nabla_\lambda S^\lambda{}_\nu - N\mathcal{K}p_\nu + Nn^\lambda\nabla_\lambda p_\nu + Nn_\nu\nabla_\lambda p^\lambda + \\
&\quad + Np^\lambda\nabla_\lambda n_\nu - N\epsilon\mathcal{K}n_\nu + \epsilon Nn^\lambda\nabla_\lambda n_\nu + n_\nu n^\lambda\nabla_\lambda \epsilon \} \\
&= -NS^\lambda{}_\nu\nabla_\lambda n^\nu - p_\nu Nn^\lambda\nabla_\lambda n^\nu - \\
&\quad - N\nabla_\lambda p^\lambda + N\epsilon\mathcal{K} - Nn^\lambda\nabla_\lambda \epsilon
\end{aligned} \tag{4.18}$$

We notice, that extended spatial quantities, e.g. the stress tensor, can be identified with the corresponding 3-quantities at the time  $t$ . Due to that, the essential projectors are present but “hidden” in them. Hence it follows

$$\begin{aligned}
0 &= NS^i{}_j K_i{}^j - ND_i p^i - 2p^i D_i N + N\epsilon\mathcal{K} - Nn^\lambda\nabla_\lambda \epsilon \\
&= NS^i{}_j K_i{}^j - 2p_i D^i N - ND_i p^i + N\epsilon\mathcal{K} - \xi^\lambda\nabla_\lambda \epsilon + \beta^\lambda\nabla_\lambda \epsilon,
\end{aligned} \tag{4.19}$$

where we have again used Eq.(3.49) but this time for the momentum density as well as Eq.(2.71).

In the case of a scalar field like  $\epsilon$ , the Lie derivative reduces to the directional derivative. Additionally using

$$\dot{\epsilon} = \mathcal{L}_\xi \epsilon \quad (4.20)$$

directly gives the evolution equation of the energy density  $\epsilon$ :

$$(\partial_t - \mathcal{L}_\beta)\epsilon = NS^i_j K_i^j + N\mathcal{K}\epsilon - ND_i p^i - 2p_i D^i N. \quad (4.21)$$

*Remark:* As in the case of the velocity field  $\mathbf{v}$ , we could also have chosen Eq.(4.20) as the starting point to derive the evolution equation. Unfortunately, this way also turns out to be very cumbersome since all kind of thermodynamical relations have to be used. Therefore, we have decided to use the way presented above. Further it should be mentioned, that this evolution equation really corresponds to the classical continuity equation. It is not obvious because we have not expressed the stress-tensor  $\mathbf{S}$  in terms of the velocity  $\mathbf{v}$ . This can easily be done by inserting Eq.(4.14) and Eq.(4.15) into Eq.(4.21).

After having made use of the energy conservation law, the momentum conservation is left. From this, the evolution equations for the velocity field can be deduced. Henceforth, we apply the orthogonal projection to the energy-momentum conservation equations. As we know that the Lie derivative must appear in order to obtain a time derivative, we again multiply these equations with  $N$  and therefore are able to reconstruct the evolution vector as needed. Starting with Eq.(4.17) we obtain

$$0 = \varphi_{,i}^\nu \{ N \nabla_\lambda S^\lambda_\nu - N \mathcal{K} p_\nu + N n^\lambda \nabla_\lambda p_\nu + N n_\nu \nabla_\lambda p^\lambda + N p^\lambda \nabla_\lambda n_\nu - N \epsilon \mathcal{K} n_\nu + \epsilon N n^\lambda \nabla_\lambda n_\nu + n_\nu N n^\lambda \nabla_\lambda \epsilon \}. \quad (4.22)$$

Making use of the orthonormality condition cancels two terms and therefore leads to

$$N \varphi_{,i}^\nu \nabla_\lambda S^\lambda_\nu - N p_i \mathcal{K} + N \varphi_{,i}^\nu n^\lambda \nabla_\lambda p_\nu + \epsilon D_i N - N p^j K_{ji} = 0. \quad (4.23)$$

Obviously, there remain only two more terms which cannot be interpreted a priori and therefore have to be treated in greater detail. Let us start with the very first term in Eq.(4.23). Making use of the relation between the covariant derivative associated with the spacetime metric  $\mathbf{g}$  and the one associated with  $\mathbf{h}$  and Eq.(2.71) yields immediately

$$N \varphi_{,i}^\nu \nabla_\lambda S^\lambda_\nu = N D_j S^j_i + S^j_i D_j N. \quad (4.24)$$



In order to interpret the third term in Eq.(4.23) some more effort has to be made. Expressing  $Nn^\lambda$  as  $\xi^\lambda - \beta^\lambda$  gives

$$\begin{aligned}
N\varphi_{,i}^\nu n^\lambda \nabla_\lambda p_\nu &= \varphi_{,i}^\nu (\xi^\lambda - \varphi_{,k}^\lambda \beta^k) \nabla_\lambda p_\nu \\
&= \varphi_{,i}^\nu \xi^\lambda \nabla_\lambda p_\nu - \beta^j D_j p_i \\
&= \varphi_{,i}^\nu \mathcal{L}_\xi p_\nu - \varphi_{,i}^\nu p_\lambda \nabla_\nu \xi^\lambda - \beta^j D_j p_i \\
&= \varphi_{,i}^\nu \mathcal{L}_\xi p_\nu - \varphi_{,i}^\nu n^\lambda p_\lambda \nabla_\nu N - \varphi_{,i}^\nu N p_\lambda \nabla_\nu n^\lambda - \\
&\quad - \varphi_{,i}^\nu p_\lambda \nabla_\nu \beta^\lambda - \beta^j D_j p_i \\
&= \varphi_{,i}^\nu \mathcal{L}_\xi p_\nu + N p_j K_i^j - p_j D_i \beta^j - \beta^j D_j p_i. \tag{4.25}
\end{aligned}$$

Additionally, we have used the definition of the Lie derivative and the orthogonality between the unit normal and the momentum co-vector as well as Eq.(2.23).

Due to previous considerations it is easily understood that the time derivative of the momentum co-vector is given by

$$\dot{p}_i = \varphi_{,i}^\nu \mathcal{L}_\xi p_\nu. \tag{4.26}$$

Besides, it follows from the definition of the Lie derivative of a co-vector along a vector that the last two terms in Eq.(4.25) exactly reproduce the Lie derivative of the momentum co-vector along the shift vector  $\beta$ .

Inserting the two auxiliary equations Eq.(4.24) and Eq.(4.25) into the intermediate result Eq.(4.23) gives

$$\begin{aligned}
0 &= N D_j S_i^j + S_i^j D_j N - N \mathcal{K} p_i + \epsilon D_i N - \\
&\quad - N p^j K_{ji} + \dot{p}_i - \mathcal{L}_\beta p_i + N p_j K_i^j. \tag{4.27}
\end{aligned}$$

The two terms containing the extrinsic curvature vanish. This yields the evolution equation for the momentum density:

$$(\partial_t - \mathcal{L}_\beta) p_i = N p_i \mathcal{K} - N D_j S_i^j - S_i^j D_j N - \epsilon D_i N. \tag{4.28}$$

These equations are already equivalent to the Euler equations but it is not common to formulate them in terms of the momentum density. It is rather usual to use the velocity field  $\mathbf{v}$ . Therefore, we have to insert Eq.(4.15) into the above relation leading to

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)[(\epsilon + \bar{p})v_i] &= N \mathcal{K} (\epsilon + \bar{p})v_i - N D_j [(\epsilon + \bar{p})v_i v^j + \bar{p} \delta_i^j] - \\
&\quad - \epsilon D_i N - [(\epsilon + \bar{p})v^j v_i + \bar{p} \delta_i^j] D_j N \tag{4.29}
\end{aligned}$$

The term containing the time evolution of the energy density can be expressed by the right-hand side of Eq.(4.21). Therefore we get

$$\begin{aligned}
0 = & N v_i S^m_n K_m^n + v_i N \epsilon \mathcal{K} - N_i D_j p^j - 2 v_i p_j D^j N + \\
& + v_i (\partial_t - \mathcal{L}_\beta) \bar{p} + (\epsilon + \bar{p}) (\partial_t - \mathcal{L}_\beta) v_i + \\
& + N v_i v^j D_j (\epsilon + \bar{p}) + (\epsilon + \bar{p}) N v_i D_j v^j + \\
& + (\epsilon + \bar{p}) N v^j D_j v_i + N D_i \bar{p} + (\epsilon + \bar{p}) v_i v^j D_j N + \\
& + (\epsilon + \bar{p}) D_i N - N \mathcal{K} (\epsilon + \bar{p}) v_i.
\end{aligned} \tag{4.30}$$

Taking a close look at the bracket-expressions and inserting the explicit form of the stress-tensor as given by Eq.(4.14) we see, that several terms cancel out such that the 3 + 1 Euler equations remain ultimately:

$$\begin{aligned}
& (\partial_t - \mathcal{L}_\beta) v_i + N v^j D_j v_i + N K_m^n v^m v_n v_i = \\
& = - \frac{1}{(\epsilon + \bar{p})} \{ N D_i \bar{p} + v_i (\partial_t - \mathcal{L}_\beta) \bar{p} \} + v_i v_j D^j N - D_i N.
\end{aligned} \tag{4.31}$$

*Remark:* The above result represents the 3 + 1 Euler equations in the Eulerian approach which is obvious due to the appearance of the convective derivative  $(v \cdot D)v$ .

The original Euler system contained three kinematic and two thermodynamical quantities, hence relativistic fluid dynamics is a theory of these five dependent function. Counting the equations established by now we see immediately, that we have gained only four of the five equations: one for the energy density and three for the velocity field. In order to close the system of differential equations, another one is still missing: the particle number density conservation. Only then the closed 3 + 1 Euler system is complete.

The calculation to derive the evolution equation for the particle number density in order to close the system is straightforward. We start with Eq.(4.9) multiplied by the lapse  $N$ :

$$\begin{aligned}
N \nabla_\mu (\bar{n} u^\mu) &= N \nabla_\mu [\bar{n} \gamma (n^\mu + v^\mu)] \\
&= N \nabla_\mu (n n^\mu + n v^\mu) \\
&= N n^\mu \nabla_\mu n + N n \nabla_\mu n^\mu + N n \nabla_\mu v^\mu + N v^\mu \nabla_\mu n \\
&= 0.
\end{aligned} \tag{4.32}$$

We further make use of Eq.(2.71) to derive a relation for the divergence of the purely spatial vector  $\mathbf{v}$ , yielding:

$$N D_i v^i \equiv N D_\mu v^\mu = N \nabla_\mu v^\mu - v^\sigma N n^\mu \nabla_\mu n_\sigma = N \nabla_\mu v^\mu - v^i D_i N. \tag{4.33}$$

Inserting the above relation into Eq.(4.32) then gives

$$N n^\mu \nabla_\mu n - N \mathcal{K} n + D_i (N n v^i) = 0. \tag{4.34}$$

Since we have multiplied the equation by  $N$ , the product  $Nn^\lambda$  can be expressed in terms of the time evolution vector field and the shift:

$$\xi^\mu \nabla_\mu n - \beta^\mu \nabla_\mu n - N\mathcal{K}n + ND_i(Nnv^i) = 0. \quad (4.35)$$

Since the derivative of a scalar field in the direction of  $\xi$  corresponds to its time derivative, we finally get the *evolution equation for the particle number density*:

$$(\partial_t - \mathcal{L}_\beta)n = N\mathcal{K}n - D_i(Nnv^i). \quad (4.36)$$

We conclude that for a given equation of state the  $3 + 1$  Euler system is given completely by the equations (4.21), (4.31) and (4.36). These equations coincide with the ones found by [20] in the case of a regular foliation.

*Remark:* As in the case of the Einstein field equations and the Maxwell equations, the general relativistic Euler initial problem is also subject to a constraint equation, namely to Eq.(4.3). The constraint condition  $u^\mu u_\mu + 1 = 0$  has to be satisfied all times  $t$ . With our choice of variables this constraint is taken care of automatically, since  $v_\mu u^\mu = -\gamma$  is not among our dependent variables.

The last thing remaining is to show that the new  $3 + 1$  Euler system Eq.(4.21), Eq.(4.31) and Eq.(4.36) is indeed equivalent to the original one Eq.(4.9) and Eq.(4.11). By counting equations we see that we have certainly obtained the right number of equations: the original system consists of five equations. We also gained five equations from the splitting. But in order to prove this rigorously, one would have to redo the calculation starting with the  $3 + 1$  system and deducing the original general relativistic Euler equations encoded in the divergence of the energy-stress tensor and the particle number conservation law.

**Lemma 2** *The  $3 + 1$  Euler system Eq.(4.21), Eq.(4.31) and Eq.(4.36) is equivalent to the original Euler equations given by Eq.(4.9) and Eq.(5.67).*

**Proof:** *We don't give the full proof here but sketch it. Starting with the evolution equation for the energy density, one can reconstruct the stress-energy tensor of the ideal fluid by making use of Eq.(4.31). Once we have reconstructed the stress-energy tensor, it is easy to show that it is divergence-free due to the Bianchi identity.*

*It is even easier to prove the equivalence of the particle number conservation and the evolution equation for  $n$ , since one only has redo all the steps of calculation.*

□

## 4.4 The Taub formulation

So far, we concentrated on the standard formulation of Euler's equations, where the velocity field  $\mathbf{v}$  is the main dynamical quantity. It followed that the entire fluid dynamics can then be described via the conservation laws for the energy density, the momentum density and the particle number density for a given equation of state. Nonetheless, it turns out that it is often more use- and fruitful to work with a different dynamical quantity, namely the **vorticity field** instead [12]. The main reason for the change of the dynamical variable is that the governing equations of motion turn out to be much simpler than in the velocity domain. Further, many flows are characterised by local regions of intense rotation, e.g. smoke rings, hurricanes or the great red spot on Jupiter. Therefore it is necessary to study the vorticity of the flow. In non-relativistic hydrodynamics the vorticity is simply:

**Definition 16** *The **vorticity field**  $\omega$  is defined as the curl of the velocity field  $\mathbf{v}$ :*

$$\omega := \text{curl } \mathbf{v} = \mathbf{D} \times \mathbf{v}. \quad (4.37)$$

Its relativistic counterpart turns out to be an antisymmetric tensor of second rank, the so-called **vorticity tensor**  $\Omega$ . This generalisation of the vorticity field was found by Lichnerowicz and Taub [28, 38]. In order to derive the reformulation of the Euler equations in terms of the vorticity tensor they used a *necessary assumption*: the equation of state has to be **barotropic**.

**Definition 17** *A **barotropic fluid** is characterised by an equation of state such that the pressure  $p$ <sup>1</sup> only depends on the energy density, hence*

$$p = p(\epsilon), \quad (4.38)$$

where the energy density  $\epsilon$  itself is a function of the particle number density  $n$  and the specific entropy  $s$ . Thus, the EOS can equally be assumed to be dependent on  $s$  and  $n$ .

*Remark:* Barotropic fluids are important idealised fluids in Astrophysics. There, the most common class of barotropic models are so-called *polytropes*.

More precisely, the equation of state of a barotropic fluid takes a form such that the energy density is a function of the product  $n \cdot m$ , where  $m$  is a  $\mathcal{C}^1$ -function in  $s$  [10]. Thus we have

$$\epsilon = \epsilon(n \cdot m(s)). \quad (4.39)$$

---

<sup>1</sup>Notice that we drop the bar in this section: we denote the pressure measured in the rest frame of the fluid by  $p$  only.

If we assume a barotropic EOS, due to the first law of thermodynamics Eq.(4.5) the differential of the energy density  $\epsilon$  is then given by

$$d\epsilon = \frac{(\epsilon + p)}{n} dn + nT ds. \quad (4.40)$$

Hence, given the EOS, one can immediately calculate

$$p(n, s) = n \left( \frac{d\epsilon}{dn} \right) \Big|_s - \epsilon(n, s) \quad T(n, s) = \frac{1}{n} \left( \frac{d\epsilon}{ds} \right) \Big|_s \quad (4.41)$$

Now, inserting the EOS given by Eq.(4.39) into the first relation in Eq.(4.41) we get

$$p = nm \frac{d\epsilon}{d(nm)} - \epsilon(nm) = p(\epsilon). \quad (4.42)$$

Let us rename our variables for reasons of convenience:  $x := nm$ ,  $y := \epsilon$  and  $g := p$ . This yields an ordinary differential equation for the variable  $x$ , which can be solved by separation:

$$xy'(x) - y(x) = g(y(x)) \Rightarrow \frac{dx}{x} = \frac{dy}{g(y) + y} \Rightarrow \ln x = \int \frac{dy}{g(y) + y} \quad (4.43)$$

Hence, the explicit expression for the product  $n \cdot m(s)$  is then

$$n \cdot m(s) = \exp \left\{ \int \frac{d\epsilon}{\epsilon + p(\epsilon)} \right\}. \quad (4.44)$$

Therefore we conclude, that any equation of state can be brought in the form of Eq.(4.39) by solving the integral Eq.(4.44).

Since  $T$  is the absolute temperature, it is a positive function. This implies that the function  $m(s)$  is strictly increasing. If further the pressure function  $p(\epsilon)$  is increasing such that the inverse exists and if also the integral

$$\int_0^p \frac{dp'}{\epsilon + p'} = f(p) \quad (4.45)$$

exists, then the energy and momentum conservation laws respectively the Euler equations can be expressed in terms of a future directed timelike vector field [10]:

$$V_\mu = F u_\mu, \quad (4.46)$$

where  $\mathbf{u}$  is the 4-velocity of the fluid and  $F := e^f \geq 1$ . This vector field is often referred to as the **Taub current**. It plays an important role in the description of vorticity in relativistic hydrodynamics and can be interpreted as the *4-momentum per particle* [26].

**Definition 18** The *vorticity tensor*  $\Omega$  is defined as

$$\Omega_{\mu\nu} := \nabla_\nu V_\mu - \nabla_\mu V_\nu. \quad (4.47)$$

*Remark:* The above defined vorticity tensor vanishes in the case of an irrotational fluid [38]. Then the vector field  $\mathbf{V}$  turns out to be the gradient of a potential and can thus be written as  $V^\mu = -g^{\mu\nu}\partial_\nu\phi$  [10].

We further note, that an **isentropic**<sup>2</sup> ideal fluid is also barotropic. An isentropic flow is defined by

$$u^\mu\nabla_\mu s = 0. \quad (4.48)$$

This together with the particle number conservation implies the following relation:

$$\nabla_\mu(snu^\mu) = 0, \quad (4.49)$$

where  $s \cdot n$  denotes the total entropy density and  $snu^\mu$  is often called the entropy current density. Again notice that  $n$  is the number density measured in the MCRF of the fluid. We have again dropped the bar in the notation. This is important, since in order to establish the Euler equation in terms of the vorticity tensor, it follows from the variational principle of hydrodynamics that the particle number conservation and the conservation of entropy have to be imposed [8]. Therefore, the consideration of an barotropic fluid is necessary. This finally yields the Euler equations in the variable  $\Omega$  respectively in  $\mathbf{V}$ :

$$\Omega_{\mu\nu}u^\nu = T\nabla_\mu s. \quad (4.50)$$

*Remark:* We do not provide the proof of the equivalence of the standard formulation and this one here. The proofs can for example be found in the papers by Taub and Brown [38, 8].

Our aim is to derive the 3 + 1 form of the Euler equations in their new form. Therefore, we have to apply both projection opportunities, the orthogonal projection provided by the pullback and the normal projection, yielding

$$Nn^\mu\Omega_{\mu\nu}u^\nu = Nn^\mu T\nabla_\mu s, \quad (4.51)$$

$$\varphi_{,i}^\mu\Omega_{\mu\nu}u^\nu = \varphi_{,i}^\mu T\nabla_\mu s. \quad (4.52)$$

Since in both equations the term  $\Omega_{\mu\nu}u^\nu$  appears, it is our first goal to simplify this expression in terms of the Taub current. Inserting the expressions for the 4-velocity and the Taub current then gives:

$$\begin{aligned} \Omega_{\mu\nu}u^\nu &= \gamma(n^\nu + v^\nu)\{\nabla_\nu V_\mu - \nabla_\mu V_\nu\} \\ &= \gamma(n^\nu + v^\nu)\{F\nabla_\nu u_\mu + u_\mu\nabla_\nu F - F\nabla_\mu u_\nu - u_\nu\nabla_\mu F\} \\ &= \gamma(n^\nu + v^\nu)\{(n_\mu + v_\mu)\nabla_\nu(F\gamma) - (n_\nu + v_\nu)\nabla_\mu(F\gamma) + \\ &\quad + F\gamma\nabla_\nu n_\mu + F\gamma\nabla_\nu v_\mu - F\gamma\nabla_\mu n_\nu - F\gamma\nabla_\mu v_\nu\}. \end{aligned} \quad (4.53)$$

---

<sup>2</sup>Isentropic means adiabatic and reversible.

In order to establish the final expression we have taken into account Eq.(2.17), Eq.(2.16) and the fact that the normal is orthogonal to the spatial velocity  $\mathbf{v}$ .

After this useful simplification, we can now start with the first projection. Therefore, we apply  $N\mathbf{n}$  to the above equation:

$$\begin{aligned}
Nn^\mu\Omega_{\mu\nu}u^\nu &= -N\gamma(n^\nu + v^\nu)\nabla_\nu(F\gamma) + \frac{1}{\gamma}Nn^\mu\nabla_\mu(F\gamma) + \\
&\quad + F\gamma Nn^\mu u^\nu\nabla_\nu n_\mu + F\gamma Nn^\mu u^\nu\nabla_\nu v_\mu - \\
&\quad - F\gamma u_\nu Nn^\mu\nabla_\mu n^\nu - F\gamma u^\nu Nn^\mu\nabla_\mu v_\nu \\
&= -N\gamma(n^\nu + v^\nu)\nabla_\nu(F\gamma) + \frac{1}{\gamma}Nn^\mu\nabla_\mu(F\gamma) - \\
&\quad - F\gamma^2 v_\mu Nn^\nu\nabla_\nu n^\mu - F\gamma^2 Nv^\mu v^\nu\nabla_\nu n_\mu - \\
&\quad - F\gamma^2(n_\nu + v_\nu)\varphi_{,j}^\nu D^j N + F\gamma^2 v_\nu Nn^\mu\nabla_\mu n^\nu - \\
&\quad - F\gamma^2 v^\nu Nn^\mu\nabla_\mu v_\nu \\
&= -N\gamma(n^\nu + v^\nu)\nabla_\nu(F\gamma) + \frac{1}{\gamma}Nn^\mu\nabla_\mu(F\gamma) + \\
&\quad + F\gamma^2 Nv^\mu v^\nu K_{\mu\nu} - F\gamma^2 v_\nu\varphi_{,j}^\nu D^j N - \\
&\quad - F\gamma^2 v^\nu Nn^\mu\nabla_\mu v_\nu. \tag{4.54}
\end{aligned}$$

We have used the same relations as already mentioned below Eq.(4.53). Note that we used Eq.(2.23) since the right projector is present due to the spatial velocity  $\mathbf{v}$ . Additionally, we made use of Eq.(2.71) and

$$\gamma^2 = \frac{1}{1 - v^k v_k}. \tag{4.55}$$

The very last term on the right-hand side of Eq.(4.54) can be rewritten as  $F\gamma^2 v^\nu Nn^\mu\nabla_\mu v_\nu = \frac{1}{2\gamma^2}FNn^\mu\nabla_\mu\gamma^2$ .

Further, expressing  $Nn^\lambda$  in terms of  $\xi$  and the shift  $\beta$  gives

$$\begin{aligned}
Nn^\mu\Omega_{\mu\nu}u^\nu &= \frac{1 - \gamma^2}{\gamma}(\xi^\mu - \beta^\mu)\nabla_\mu(F\gamma) - N\gamma v^j D_j(F\gamma) + \\
&\quad + NF\gamma^2 v^j v^i K_{ij} - \frac{1}{2\gamma^2}F(\xi^\mu - \beta^\mu)\nabla_\mu\gamma^2 - F\gamma^2 v^j D_j N \tag{4.56}
\end{aligned}$$

Until now we have only considered the left-hand side of Eq.(4.50). Applying  $Nn^\mu$  to right-hand side gives

$$\begin{aligned}
TNn^\mu\nabla_\mu s &= T(\xi^\mu - \beta^\mu)\nabla_\mu s \\
&= T(\mathcal{L}_\xi - \mathcal{L}_\beta)s \\
&= T(\partial_t - \mathcal{L}_\beta)s, \tag{4.57}
\end{aligned}$$

since  $s$  is only a scalar function and not a vector field. Let us further introduce the *spatial Taub current* by

$$V_i = \varphi_{,i}^\mu V_\mu = F\gamma v_i. \quad (4.58)$$

Using again the definition of the Lie derivative of a function and Eq.(4.58) the first evolution equation finally writes as:

$$\begin{aligned} T(\partial_t - \mathcal{L}_\beta)s = & -\gamma v^k v_k (\partial_t - \mathcal{L}_\beta)(F\gamma) - N\gamma v^j D_j(F\gamma) - \\ & + N\gamma V^j v^i K_{ij} - \gamma V^j D_j N - \frac{1}{2\gamma^2} F(\partial_t - \mathcal{L}_\beta)\gamma^2. \end{aligned} \quad (4.59)$$

It describes the evolution of the specific entropy  $s$ .

In order to gain the equivalent 3 + 1 form of Eq.(4.50) the orthogonal projection has to be performed as well. Thus, for the left-hand side given by Eq.(4.53) we obtain:

$$\begin{aligned} \varphi_{,i}^\mu \Omega_{\mu\nu} u^\nu = & \varphi_{,i}^\mu u^\nu \{ (n_\mu + v_\mu) \nabla_\nu (F\gamma) - (n_\nu + v_\nu) \nabla_\mu (F\gamma) + \\ & + F\gamma \nabla_\nu n_\mu + F\gamma \nabla_\nu v_\mu - F\gamma \nabla_\mu n_\nu - F\gamma \nabla_\mu v_\nu \} \\ = & u^\nu v_i \nabla_\nu (F\gamma) - u^\nu (n_\nu + v_\nu) D_i (F\gamma) - u^\nu F\gamma \varphi_{,i}^\mu K_{\mu\nu} + \\ & + u^\nu \varphi_{,i}^\mu F\gamma \nabla_\nu v_\mu + u^\nu \varphi_{,i}^\mu F\gamma K_{\mu\nu} - F\gamma \varphi_{,i}^\mu u^\nu \nabla_\mu v_\nu \\ = & \gamma v_i n^\nu \nabla_\nu (F\gamma) + \gamma v_i v^k D_k (F\gamma) + \gamma (1 - v_k v^k) D_i (F\gamma) + \\ & + F\gamma^2 \varphi_{,i}^\mu n^\nu \nabla_\nu v_\mu + F\gamma^2 v^k D_k v_i + F\gamma^2 \varphi_{,i}^\mu v^\nu \nabla_\mu n_\nu - \\ & - F\gamma^2 v^k D_i v_k \\ = & \gamma v_i v^k D_k (F\gamma) + \frac{1}{\gamma} D_i (F\gamma) + F\gamma \varphi_{,i}^\mu n^\nu \nabla_\nu (F\gamma v_\mu) + \\ & + F\gamma^2 v^k (D_k v_i - D_i v_k) - F\gamma^2 K_{ij} v^j. \end{aligned} \quad (4.60)$$

The above calculation is straightforward, making use of all necessary relations, in detail already mentioned below Eq.(4.53). It is a priori not clear, how the term  $\gamma \varphi_{,i}^\mu n^\nu \nabla_\nu (F\gamma v_\mu)$  in the above equation shall be interpreted. For this reason we have to deal with it in greater detail. Since we are interested in deriving the time evolution for the spatial Taub current, we have to multiply the above relation by  $N$  in order to obtain well-defined expressions.



For the ambiguous term we then get:

$$\begin{aligned}
\gamma\varphi_{,i}^\mu N n^\nu \nabla_\nu (F\gamma v_\mu) &= \gamma\varphi_{,i}^\mu (\xi^\nu - \beta^\nu) (F\gamma v_\mu) \\
&= \gamma^2 F \varphi_{,i}^\mu \xi^\nu \nabla_\nu v_\mu + \gamma\varphi_{,i}^\mu v_\mu \xi^\nu \nabla_\nu (F\gamma) - \gamma^2 F \varphi_{,i}^\mu \beta^\nu \nabla_\nu v_\mu - \gamma\varphi_{,i}^\mu v_\mu \beta^\nu \nabla_\nu (F\gamma) \\
&= \gamma v_i \mathcal{L}_\xi (F\gamma) + \gamma^2 F \varphi_{,i}^\mu \mathcal{L}_\xi v_\mu - \gamma^2 F \varphi_{,i}^\mu v_\nu \nabla_\mu \xi^\nu - \gamma v_i \beta^k D_k (F\gamma) - \\
&\quad - \gamma^2 F \beta^k D_k v_i \\
&= \gamma \partial_t (F\gamma v_i) - \gamma^2 F v_k D_i \beta^k + \gamma^2 N F v_k K_i^k - \gamma v_i \beta^k D_k (F\gamma) - \gamma^2 F \beta^k D_k v_i \\
&= \gamma \partial_t (F\gamma v_i) + \gamma^2 N F v_k K_i^k - \gamma F \mathcal{L}_\beta v_i - \gamma v_i \mathcal{L}_\beta (F\gamma) \\
&= \gamma \partial_t (F\gamma v_i) - \gamma \mathcal{L}_\beta (F\gamma v_i) + \gamma^2 N F v_k K_i^k. \tag{4.61}
\end{aligned}$$

Reinserting this important auxiliary result in our original equation Eq.(4.60) then yields for the left-hand side the following result:

$$\begin{aligned}
N\varphi_{,i}^\mu \Omega_{\mu\nu} u^\nu &= \gamma \partial_t (F\gamma v_i) - \gamma \mathcal{L}_\beta (F\gamma v_i) + \gamma^2 N F v^k K_{ik} + \frac{N}{\gamma} D_i (F\gamma) + \\
&\quad + \gamma N v_i v^k D_k (F\gamma) + N F \gamma^2 v^k (D_k v_i - D_i v_k) - N F \gamma^2 K_{ij} v^j \\
&= \gamma (\partial_t - \mathcal{L}_\beta) V_i + \frac{N}{\gamma} D_i (F\gamma) + N \gamma v^k D_k V_i - \\
&\quad - N \gamma v^k D_i V_k + N \gamma v_k v^k D_i (F\gamma) \\
&= \gamma (\partial_t - \mathcal{L}_\beta) V_i + \gamma D_i (F\gamma) + N \gamma (v^k D_k V_i - v^k D_i V_k). \tag{4.62}
\end{aligned}$$

The projection of the right-hand side of Eq.(4.50) is calculated easily and turns out to be:

$$NT\varphi_{,i}^\mu \nabla_\mu s = NT D_i s. \tag{4.63}$$

Eventually, putting all pieces together we obtain the final time evolution equation for the Taub current:

$$(\partial_t - \mathcal{L}_\beta) V_i = \frac{NT}{\gamma} D_i s - D_i (F\gamma) - N (v^k D_k V_i - v^k D_i V_k). \tag{4.64}$$

We conclude, that in the case of a barotropic fluid, the Euler equations can be reformulated in terms of the vorticity tensor respectively the Taub current. The 3 + 1 system is then given by the equations Eq.(4.59) and Eq.(4.64).

## Chapter 5

# The 3 + 1 Formulation of the Einstein Equations

Since the last chapters focused entirely on geometric concepts and properties as well as on the 3 + 1 formulation of electromagnetism and ideal hydrodynamics on a curved background, we now turn to full General Relativity: it is time to apply the 3 + 1 splitting to the fundamental equations of General Relativity: *Einstein's field equations*. This procedure allows us to deduce a valid initial value problem formulation of General Relativity itself. In order to do so, we consider a 4-dimensional semi-Riemannian spacetime  $(M, g)$  such, that the spacetime metric  $\mathbf{g}$  fulfills the Einstein equations with a vanishing cosmological constant  $\Lambda$ :

$$\mathbf{G} := {}^4\mathbf{R} - \frac{1}{2} {}^4\mathcal{R} \mathbf{g} = 8\pi \mathbf{T}, \quad (5.1)$$

where  $\mathbf{G}$  is the Einstein tensor,  ${}^4\mathbf{R}$  the Ricci tensor associated with the spacetime metric  $\mathbf{g}$  as given by Eq.(2.7),  ${}^4\mathcal{R}$  the corresponding Ricci scalar and  $\mathbf{T}$  is the stress-energy tensor of the source of the gravitational field (for further details see *Appendix D*). Note that the right-hand side vanishes in the case of vacuum, hence the vacuum field equations reduce to  $\mathbf{G} = 0$ . In components, i.e. in local coordinates  $\{x^\sigma\}$  on  $M$ , Eq.(5.1) writes as

$$G_{\mu\nu} := {}^4R_{\mu\nu} - \frac{1}{2} {}^4\mathcal{R} g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (5.2)$$

In order to obtain the 3 + 1 version of Eq.(5.2) one could either apply the orthogonal projection introduced in Sec. 2.3 directly to the Einstein tensor, or to the equivalent form of the field equations given by

$${}^4R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2} \mathcal{T} g_{\mu\nu}), \quad (5.3)$$

where  $\mathcal{T}$  is the trace of the stress-energy tensor with respect to the spacetime metric  $\mathbf{g}$ . It turns out that the projection of Eq.(5.3) is more handy, thus we

follow the approach presented in [20]. The “direct splitting” of the standard form of Einstein’s equations can be found in [42].

## 5.1 Projection of Einstein’s equations

We consider a Lorentzian spacetime and a spacelike hypersurface  $\varphi(\Sigma)$  fulfilling all properties given in Sec. 2.1. Hence, the approach is now to divide Einstein’s equations into parts tangent to the hypersurface and orthogonal to it. Further we assume a family of smooth embeddings generated by a local vector field  $\xi(x)$  on  $M$ . Therefore, the concepts of the orthogonal projection, the unit normal and further the time evolution play a crucial role (see Sec. 2.3).

Let us start with the orthogonal projection of the left-hand side of Eq.(5.3), which we have multiplied by  $N$  to make use of Eq.(2.74). Due to the appearance of successive covariant derivatives it is far more complicated to derive the final result if we stick to a general slicing. Therefore, we assume  $N > 0$  for this derivation only. If we do so, we are able to use the relation Eq.(2.26) which simplifies the derivation significantly. Further, taking into account the equations (2.54) as well as the orthonormality condition  $\varphi_{,i}^\sigma n_\sigma = 0$  we obtain:

$$\begin{aligned}
N\varphi_{,i}^\mu\varphi_{,j}^\nu{}^4R_{\mu\nu} &= N^3R_{ij} + N\mathcal{K}K_{ij} - NK_{im}K_j{}^m - N\varphi_{,i}^\mu\varphi_{,j}^\nu{}^4R_{\mu\sigma\nu\rho}n^\sigma n^\rho \\
&= N^3R_{ij} + N\mathcal{K}K_{ij} - NK_{im}K_j{}^m - \\
&\quad - N\varphi_{,i}^\mu\varphi_{,j}^\nu n^\rho(\nabla_\nu\nabla_\rho - \nabla_\rho\nabla_\nu)n_\mu \\
&= N^3R_{ij} + N\mathcal{K}K_{ij} - NK_{im}K_j{}^m - \\
&\quad - N\varphi_{,i}^\mu\varphi_{,j}^\nu n^\rho[\nabla_\nu(-K_{\rho\mu} - a_\mu n_\rho) - \nabla_\rho(-K_{\mu\nu} - a_\nu n_\mu)] \\
&= N^3R_{ij} + N\mathcal{K}K_{ij} - NK_{im}K_j{}^m - N\varphi_{,i}^\mu\varphi_{,j}^\nu(K_{\rho\mu}\nabla_\nu n^\rho + \\
&\quad + \nabla_\nu a_\mu + n^\rho\nabla_\rho K_{\mu\nu} + a_\mu a_\nu) \\
&= N^3R_{ij} + N\mathcal{K}K_{ij} - D_i D_j N - N\varphi_{,i}^\mu\varphi_{,j}^\nu n^\rho\nabla_\rho K_{\mu\nu}, \quad (5.4)
\end{aligned}$$

where we have also used  $N\nabla_\nu a_\mu = \nabla_\nu(Na_\mu) - a_\mu\nabla_\nu N$  to get the last line of Eq.(5.4).

Let us now show that the very last term in Eq.(5.4) is related to the time evolution of the extrinsic curvature  $\mathbf{K}$ . The Lie derivative of  $\mathbf{K}$  along the time evolution vector field  $\xi$  according to *Appendix A* is given by

$$\mathcal{L}_\xi K_{\mu\nu} = \xi^\rho\nabla_\rho K_{\mu\nu} + K_{\mu\rho}\nabla_\nu\xi^\rho + K_{\rho\nu}\nabla_\mu\xi^\rho. \quad (5.5)$$

Comparison with Eq.(5.4) and making use of  $Nn^\rho = \xi^\rho - \beta^\rho$ , yields the following intermediate result:

$$\begin{aligned}
N\varphi_{,i}^\mu\varphi_{,j}^\nu n^\rho \nabla_\rho K_{\mu\nu} &= \varphi_{,i}^\mu\varphi_{,j}^\nu (\mathcal{L}_\xi K_{\mu\nu} - K_{\mu\rho}\nabla_\nu \xi^\rho - K_{\rho\nu}\nabla_\mu \xi^\rho - \beta^\rho \nabla_\rho K_{\mu\nu}) \\
&= \dot{K}_{ij} - \varphi_{,i}^\mu\varphi_{,j}^\nu (NK_{\mu\rho}\nabla_\nu n^\rho - K_{\mu\rho}\nabla_\nu \beta^\rho - \\
&\quad - NK_{\rho\nu}\nabla_\mu n^\rho - K_{\rho\nu}\nabla_\mu \beta^\rho - \beta^\rho \nabla_\rho K_{\mu\nu}) \\
&= \dot{K}_{ij} - \varphi_{,i}^\mu\varphi_{,j}^\nu \mathcal{L}_\beta K_{\mu\nu} + 2NK_{im}K_j^m, \tag{5.6}
\end{aligned}$$

where we have taken into account that  $K_{\alpha\beta}n^\beta = 0$  since  $\mathbf{K}$  is tangential to the hypersurface.

Now, taking the above relation and inserting it into Eq.(5.4) leads to the designated final result for the left-hand side of the field equations:

$$N\varphi_{,i}^\mu\varphi_{,j}^\nu {}^4R_{\mu\nu} = N^3R_{ij} + N\mathcal{K}K_{ij} - \dot{K}_{ij} + \mathcal{L}_\beta K_{ij} - 2NK_{im}K_j^m - D_i D_j N \tag{5.7}$$

The afore derived equation is only the first part of the projection of the Einstein equations onto  $\Sigma$ . Now we have to work out the right-hand side as well which is done very easily:

$$8\pi N\varphi_{,i}^\mu\varphi_{,j}^\nu (T_{\mu\nu} - \frac{1}{2}{}^4\mathcal{T}g_{\mu\nu}) = 8\pi N(S_{ij} - \frac{1}{2}{}^4\mathcal{T}h_{ij}), \tag{5.8}$$

where  $S_{ij}$  denotes the *stress tensor*. All together this gives now

$$N^3R_{ij} + N\mathcal{K}K_{ij} - \dot{K}_{ij} + \mathcal{L}_\beta K_{ij} - 2NK_{im}K_j^m - D_i D_j N = 8\pi N(S_{ij} - \frac{1}{2}{}^4\mathcal{T}h_{ij}). \tag{5.9}$$

An alternative derivation of the time evolution equation for the extrinsic curvature is performed in *Appendix C*. We have chosen to more convenient way here because it is more obvious that the evolution equations for  $\mathbf{K}$  follow directly from the Einstein field equations.

*Remark:* For simplicity we have derived the evolution equation using the assumption  $N > 0$ . We state without proof that the equations remain valid in the setting of a general slicing.

Since the extrinsic curvature is a symmetric, purely spatial tensor of second rank, the above relation yields equations for three components. Together with the evolution equations for the components of the spatial metric we have a total of six equations. Einstein's field equations form a system of ten quasilinear PDEs. Thus, in order to close the system there are still four equations missing. But, since the evolution equations have already been deduced, these additional equations will rather be *constraints* which have to be imposed on the dynamical variables  $\mathbf{h}$  and  $\mathbf{K}$ .

After having performed the full projection of the Einstein equations onto  $\Sigma$ , there remain two possible projections: first we are to find the projection perpendicular to the considered hypersurface. This is done by contracting Eq.(5.2) twice with the unit normal  $\mathbf{n}$  since Einstein's equations are a tensor identity between bilinear forms. Using the normalisation of  $\mathbf{n}$ , we get

$${}^4R_{\mu\nu}n^\mu n^\nu + \frac{1}{2}{}^4\mathcal{R} = 8\pi T_{\mu\nu}n^\mu n^\nu. \quad (5.10)$$

Now, taking into account the scalar Gauß equation Eq.(2.55), we find immediately:

$${}^3\mathcal{R} + \mathcal{K}^2 - K_{ij}K^{ij} = 16\pi\epsilon, \quad (5.11)$$

where we have used that the double contraction of the stress-energy tensor  $\mathbf{T}$  with a unit normal vector gives the total energy density  $\epsilon$  of the matter w.r.t.  $\mathbf{n}$  (see *Appendix D*). Notice that the above equation does not contain any time derivatives and therefore is not an evolution equation but really a **constraint equation** as mentioned before that must be satisfied at all times  $t$ . Eq.(5.11) is often referred to as the *Hamiltonian constraint* equation since it contains the energy density.

Together with this constraint equation, we have a total of seven equations. Thus, there are still three components missing. In order to reproduce the full system, we have to perform the so-called mixed projection of the field equations:

$$\varphi_{,i}^{\mu 4}R_{\mu\nu}n^\nu - \frac{1}{2}\varphi_{,i}^{\mu 4}\mathcal{R}g_{\mu\nu}n^\nu = 8\pi\varphi_{,i}^\mu T_{\mu\nu}n^\nu. \quad (5.12)$$

The second term vanishes identically since  $\varphi_{,i}^\mu g_{\mu\nu}n^\nu = \varphi_{,i}^\mu n_\mu \equiv 0$ . Taking into account that the contraction of the stress-energy tensor with a unit normal and its pullback onto  $\Sigma$  gives minus the momentum density  $\mathbf{p}$  of the matter as measured by an observer with 4-velocity  $\mathbf{n}$  (see Eq.(D.2)) and the contracted Codazzi equation Eq.(2.45) we maintain the three missing equations to close the Einstein system:

$$D_j K_i^j - D_i \mathcal{K} = 8\pi p_i. \quad (5.13)$$

As before there are no time derivatives contained in the afore derived equation. Therefore they are also constraint equations that have to be fulfilled at all times. The above equation is known as the *momentum constraint* for obvious reasons. The constraint equations are non-linear elliptic partial differential equations for the initial data  $\mathbf{h}$  and  $\mathbf{K}$ .

We conclude that the Einstein system can be reformulated as a *constrained Cauchy problem* for the two dynamical variables, the spatial metric and the extrinsic curvature, when they are specified on an initial hypersurface.

## 5.2 The 3+1 Einstein system

Now, collecting the above results the final 3 + 1 *Einstein system* rewrites as a system of six evolution equations and four constraint equations. Thus, the general relativistic Cauchy problem is given by:

*The evolution equations:*

$$(\partial_t - \mathcal{L}_\beta)h_{ij} = -2NK_{ij} \quad (5.14)$$

$$(\partial_t - \mathcal{L}_\beta)K_{ij} = N^3R_{ij} + N\mathcal{K}K_{ij} - 2NK_{im}K_j{}^m - D_iD_jN - 8\pi N\left\{S_{ij} - \frac{1}{2}{}^4\mathcal{T}h_{ij}\right\} \quad (5.15)$$

and the *constraint equations*:

$${}^3\mathcal{R} + \mathcal{K}^2 - K_{ij}K^{ij} = 16\pi\epsilon \quad (5.16)$$

$$D_jK_i{}^j - D_i\mathcal{K} = 8\pi p_i \quad (5.17)$$

If the source terms  $(\epsilon, \mathbf{p}, \mathbf{S})$  are given, the system Eq.(5.14)-Eq.(5.17) constitutes a non-linear PDE system of second order for the variables  $(\mathbf{h}, \mathbf{K}, N, \beta)$ . The evolution equations Eq.(5.14) and Eq.(5.15) are sometimes called the *ADM equations*, although the evolution system was already known before their work.

*Remark:* It is of importance to notice that we have no evolution equations for lapse and shift. These two represent the coordinate freedom in General Relativity and therefore can be chosen freely [1].

## 5.3 The Initial Value Problem in General Relativity

General Relativity states that gravitation is described by and encoded in a 4-dimensional *Lorentzian manifold*  $M$  which is mainly characterised by a symmetric second-rank covariant tensor field, the *spacetime metric*  $\mathbf{g}$ . [42] The metric is a solution of the prescribing field equations, the *Einstein equations*. These equations form a geometric system of non-linear partial differential equations (PDEs) of second order, whereas space and time play equivalent roles. Thinking about astrophysical configurations, it is natural and reasonable to ask about the evolution of the gravitational field of a given source. This is the subject of an initial value formulation. But, in the case of General Relativity, it is a priori not clear how this shall be done, since time and space build up a 4-dimensional grid [1].

Fortunately, Einstein's field equations can be reduced to a system of strictly hyperbolic PDEs, which are, from a mathematical point of view, of the same type as the wave equation. Therefore, it appears to be feasible due to the theory of hyperbolic partial differential equations to look for the development of solutions which are given on an initial spacelike hypersurface. It had been proven several decades ago that Einstein's theory of gravity can be rewritten such, that the dynamical evolution can be determined uniquely. In other words, General Relativity possesses a well-posed *initial value formulation*. Such a formulation demands a separation between "space" and "time" such, that Einstein's equations are rewritten in terms of purely spatial, time-dependent tensor quantities. The explicit performance of this reformulation of the field equations is done in the previous section. We now focus on the formulation and the properties of the initial value problem in General Relativity.

The concept of an initial value problem or Cauchy problem is of central importance in the study of differential equations. Roughly speaking, the **initial value problem** simply consists of a **differential equation** (either ordinary or partial) and **initial conditions**. In the case of second order differential equations, naturally two appropriate initial conditions have to be specified. Rigorously, the initial value problem for an ordinary differential equation is generally defined as:

**Definition 19** *Be  $k \in \mathbb{N}$  and  $f : \mathbb{R} \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$  and  $y \in \mathbb{R}^n$  a function of the independent variable  $t$ . Then, the initial value problem of  $k^{th}$ -order is given by*

$$\begin{aligned} y^{(k)} &= f(t, y(t), y'(t), \dots, y^{(k-1)}(t)) \\ y^{(i)}(t_0) &= y_i, \quad i \in \{0, 1, \dots, k-1\}. \end{aligned} \quad (5.18)$$

Note that the initial points have to lie in the domain of the function  $f$  in order to define the initial value problem. A solution of the initial value problem is a function  $y(t)$  that is a solution of the differential equation and satisfies the initial conditions. For a large class of initial value problems, the *existence* and *uniqueness* of solutions is given due to the theorem of *Picard and Lindelöf* (see any modern text on differential equations).

On the level of partial differential equations, the initial value problem is commonly referred to as the **Cauchy problem**. Like in the ordinary case, it asks for a solution to a PDE (or a system of PDEs) that satisfies certain initial conditions given on a hypersurface in the domain. The existence and uniqueness of solutions to Cauchy problems are subject to the famous theorem by *Cauchy and Kovalevskaya* (see any modern text on PDEs):

**Theorem 1** (*Cauchy-Kovalevskaya theorem*): *Let  $\{t, x^1, \dots, x^{m-1}\}$  be coordinates of  $\mathbb{R}^m$ . Consider a system of  $n$  partial differential equations for  $n$*

unknown functions  $f_1, \dots, f_n$  in  $\mathbb{R}^m$  of the form

$$\frac{\partial^2 f_i}{\partial t^2} = F_i(t, x^j; f_k; \partial_t f_k; \partial_j f_k; \partial_t \partial_j f_k; \partial_j \partial_l f_k), \quad (5.19)$$

where each  $F_i$  is an analytic function. Let  $u_i(x^j)$  and  $v_i(x^j)$  be analytic functions. Then there is an open neighbourhood  $O$  of the initial hypersurface  $\varphi_0(\Sigma)$  such that within  $O$  there exists a unique analytic solution of Eq.(5.19) such that  $f_i(0, x^j) = u_i(x^j)$  and  $\partial_t f_i(0, x^j) = v_i(x^j)$ .

Let us now turn to the nature of the Cauchy problem in General Relativity but notice that we only provide a brief overview of the problem. A detailed and rigorous explanation can for example be found in [9]. A comprehensive summary is provided by [37].

For this purpose, let us recall that any spacelike hypersurface  $\varphi(\Sigma)$  of a Lorentzian manifold  $(M, g)$  possesses an induced Riemannian metric  $\mathbf{h}$  and a second fundamental form  $\mathbf{K}$  (see *Chapter 2*). The famous Gauß and Codazzi relations then show that certain components of the Einstein tensor  $\mathbf{G}$  are determined completely by  $\mathbf{g}$  and  $\mathbf{K}$ . This fact is the first important observation concerning the nature of the initial value problem in General Relativity: these equations have to be imposed as constraints for the initial data (see previous subsections).

Secondly, we have to define a set of **initial data**. In the context of General Relativity, an initial data set is a triple  $(\Sigma, \mathbf{h}, \mathbf{K})$ , where  $\Sigma$  is an abstract 3-dimensional manifold,  $\mathbf{h}$  a Riemannian metric and  $\mathbf{K}$  a symmetric tensor of rank 2. A **development** of the initial data set is a triple  $(M, \mathbf{g}, \varphi)$ , where  $(M, g)$  is the 4-dimensional Lorentzian manifold, the spacetime, and  $\varphi$  a diffeomorphism  $\varphi : \Sigma \rightarrow \varphi(\Sigma) \subset M$ , such that  $h = \varphi^*g$  and  $\varphi K$  is the extrinsic curvature of  $\varphi(\Sigma)$  as submanifold of  $(M, g)$ . A development of  $(M, \mathbf{g}, \varphi)$  is called an **Einsteinian development** or E-development if the metric  $\mathbf{g}$  satisfies the Einstein equations on  $M$  [9].

As already mentioned before, the initial data  $(\mathbf{h}, \mathbf{K})$  cannot be chosen arbitrarily, but have to satisfy the constraint equations Eq.(5.16) and Eq.(5.17). If and only if  $\mathbf{h}$  and  $\mathbf{K}$  satisfy the constraints, an E-development can exist.

One crucial point in the discussion of the Cauchy problem is the **well-posedness** of the problem. This means that

- the problem has a solution
- the solution is unique
- the solution depends continuously on the initial data.



Moreover, according to the definition of a Cauchy problem, must the evolution equations guarantee the propagation of the constraints. If one considers the Einstein vacuum equations for reasons of simplicity and makes use of the contracted Bianchi identity, one obtains a first order linear homogeneous system of PDEs. Thus, if these equations are satisfied for  $t = 0$ , they are satisfied for all times later for given analytic data. Rigorously, this can be proven with the theorems that guarantee the existence and uniqueness of a solution to the Cauchy problem in General Relativity [37, 43].

In order to obtain at least the local existence and uniqueness theorems in General Relativity, one makes use of the harmonic gauge condition, since Einstein's equations turn out to be a strictly hyperbolic system in this gauge. This special gauge condition requires that the coordinate functions  $x^\mu$  are harmonic:

$$H^\mu \equiv \square x^\mu = \nabla_\sigma \nabla^\sigma x^\mu = 0. \quad (5.20)$$

This is of crucial importance, because there exist mathematical theorems for these kind of PDE systems, which guarantee the existence and uniqueness of solutions. Another equally important fact is that it is sufficient to solve the Cauchy problem for the reduced Einstein system since it guarantees the propagation of the harmonic gauge condition. If the gauge condition and the constraints are imposed on the initial surface, they hold for all later times. Hence, also the field equations are satisfied. In the harmonic gauge, the vacuum field equations rewrite as<sup>1</sup>:

$$\begin{aligned} 0 = R^H_{\mu\nu} &= R_{\mu\nu} + \frac{1}{2} \sum_{\alpha} (g_{\alpha\mu} \partial_\nu + g_{\alpha\nu} \partial_\mu) H^\alpha \\ &= -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g), \end{aligned} \quad (5.21)$$

where  $\hat{F}$  is a non-linear function of the metric components and their first derivatives. Thus, the original Einstein equations are equivalent to the above system of quasi-linear hyperbolic PDEs together with the harmonic gauge condition. The PDE system Eq.(5.21) is known as the “reduced Einstein equations”. The key point is, that Einstein's equations are now in such a form that well-established mathematical theorems tell us, that the equations possess locally a unique solution for smooth initial data  $g_{\mu\nu}(t = 0, x^i)$  and  $\partial_t g_{\mu\nu}(t = 0, x^i)$ . In order to obtain even global uniqueness, the considered spacetime has to be globally hyperbolic. For the relevant theorems see [42] or [9]. Therefore, we conclude that General Relativity definitely possess a valid initial value formulation in the vacuum case.

---

<sup>1</sup>The explicit and detailed derivation can be found in [42] Chapter 10.

Finally, let us make a brief comment on the well-posedness of the initial value formulation of the Einstein equations with *matter sources*. Its existence depends strongly on the dynamical equations satisfied by the matter and the explicit form of the stress-energy tensor. A well-posed initial value formulation only exists, if the matter consists of fields which satisfy a quasi-linear hyperbolic equation and if the stress-energy tensor depends only on these fields, the metric and the first derivatives of them, because then the combined Einstein system in the harmonic gauge takes again the form of Eq.(5.21). Thus, the Einstein-Maxwell equations have a well-posed initial value formulation. It was further shown that some systems, which cannot be rewritten in this form, also possess a well-posed initial value formulation, in particular the Einstein-perfect fluid system for a barotropic equation of state [24]. But the initial value formulation is not an automatic feature in most field theories and therefore certain field equations generalised to curved spacetime do not possess a valid initial value formulation [42].

## Chapter 6

# Summary

This chapter contains a brief overview of the topics treated and the results obtained in this work. All relevant formulae are derived and explained in great detail in the main text.

### *Chapter 2*

Since the knowledge of hypersurfaces is crucial for the understanding of the  $3 + 1$  formalism of General Relativity, the entire chapter is dedicated to main geometric and kinematic properties of Riemannian hypersurfaces.

First, we give an introduction to the differential geometry of a single hypersurface, later we introduce a general family of such surfaces - a so-called *slicing* - generated by a 1-parameter family of smooth embeddings, which is a far more general case than the standard one which only considers non-intersecting hypersurfaces, commonly called a foliation. The two main gauge quantities, lapse  $N$  and shift  $\beta$  are presented here. In this chapter we also define several important vector fields along the embedding such as the normal vector field, which connects the parameter derivative along the embedding with the Lie derivative in the ambient 4-dimensional Lorentzian manifold. We spend quite some time on this general formalism since we do not impose any constraints on lapse and shift and therefore many quantities are not defined on the spacetime manifold  $M$  anymore. Important relations between 3- and 4-dimensional objects like the famous Gauß and Codazzi relations are deduced. Finally, we compare the standard foliation case with a general slicing in greater detail.

Let us note that not only the fundamental geometric objects and their relations are defined but also a specific notation is developed in this chapter which is used throughout the entire work.

### *Chapter 3*

This chapter is devoted to the explicit derivation of the  $3 + 1$  equations of Maxwell's theory of electromagnetism. We assume an electromagnetic plasma to propagate on a given curved metric background. Hence, we neglect the gravity induced by them and *do not* study the Einstein-Maxwell equations. These can be found in [2].

The first part gives a brief introduction to Maxwell's theory of electromagnetism. We summarise its covariant formulation and deduce the  $3 + 1$  equations in a completely general way. We see then explicitly that the initial value problem in Maxwell's theory is subject to two constraint conditions - the Gauß law and the non-existence of magnetic monopoles.

### *Chapter 4*

After the detailed discussion of the electromagnetic field and its  $3 + 1$  decomposition, we turn to general relativistic hydrodynamics in the case of a one-component fluid. We restrict the discussion to ideal or perfect fluids, thus no heat conduction and/or viscosity are present.

First, we summarise the thermodynamics and the kinematics of a perfect fluid yielding the general relativistic form of the Euler equations represented by three conservation laws plus an equation of state. Secondly, we decompose the 4-dimensional equations yielding their final  $3 + 1$  form consisting of the evolution equations for the spatial velocity and the particle number density as well as of the normalisation condition for the 4-velocity which is again a constraint equation.

We then turn towards a different formulation of hydrodynamics, namely to the so-called *Taub formulation*. This formulation plays an important role in the context of circulation and vorticity. It is of main interest in the study of turbulent flows and convection. For this purpose, barotropic fluids and the *Taub current*  $V_\mu$  are introduced. The final  $3 + 1$  system consists of the evolution equations for the specific entropy and the spatial Taub current and again of the constraint equation for the 4-velocity.

### *Chapter 5*

In *Chapter 5* we first introduce the Einstein field equations and then deduce their equivalent  $3 + 1$  formulation. Therefore we apply the formalism developed in *Chapter 2* to them, although we assume  $N > 0$  in this derivation for reasons of simplicity. It turns out that the initial value formulation of General Relativity leads to a constrained Cauchy problem. The  $3 + 1$  form of the Einstein equations consists of three evolution equations for the spatial metric  $\mathbf{h}$ , which are purely due to the kinematics of a hypersurface, three evolution equations for the extrinsic curvature  $\mathbf{K}$  and four constraint

equations, known as the *Hamiltonian* and the *momentum constraint*. An alternative, less obvious but more elegant derivation of the evolution equations for the extrinsic curvature can be found in *Appendix C*.

After the derivation of the  $3 + 1$  Einstein system, we present a brief overview of the Cauchy problem in General Relativity. The local version of the initial value problem is explained and crucial theorems governing the existence and uniqueness of solutions are mentioned.

# Concluding Remarks

The well-posed initial value formulation of General Relativity had a major impact on the field of Numerical Relativity, leading to the development of codes which determine the motion of black holes, neutron stars, accretion discs etc. in full General Relativity and not only in post-Newtonian approximations. In order to do so, a specific slicing has to be chosen. In the standard formulation this is chosen to be a regular foliation. In our general approach to the  $3 + 1$  formulation we do not restrict the discussion to a foliation but instead try to establish meaningful expressions under the consideration of a slicing. In such a setting all relevant quantities are defined only along the given family of embeddings. Nonetheless, it is not clear yet whether the associated initial value problem is well-posed in the case of a vanishing lapse. But, as expected, we gain the same  $3 + 1$  equations as in the context of a foliation. We want to emphasise that one of the key points even in our general approach is the development of a meaningful concept of the time derivative of the normal although it is not a vector field on  $M$  anymore. This general formulation is not only interesting from a pure mathematical point of view, it might also broadens the range of relevant and applicable slicings in Numerical Relativity, especially in cosmological contexts.

Further, we provide the explicit and detailed derivation of the general  $3 + 1$  Maxwell and Euler equations. Though we have not done the full Einstein-Maxwell or Einstein-Euler equations, the combination is rather easy since we have already done one part of it. Additionally, is it now not difficult to combine the  $3 + 1$  Maxwell and  $3 + 1$  Euler equations in order to establish the general  $3 + 1$  form of magnetohydrodynamics. However, the consideration of an electromagnetic field and/or a perfect fluid in a curved background plays an important role in the description of accretion discs or matter flows and thus is relevant for numerical simulations.

Moreover, we present the  $3 + 1$  version of the Taub formulation of ideal hydrodynamics. This special formulation is of importance regarding the derivation of the equations of motion from the variational principle. The Taub formulation has recently become very popular in the context of superfluidity and the study of neutron stars [34]. The explicit and detailed

derivation of its  $3 + 1$  version might be useful for the simulation of neutron star matter in full General Relativity.

# Appendix A

## Lie Derivative

One of the most useful and essential analytic tools in differential geometry is the **Lie derivative** along a congruence. Since it is also fundamental in the  $3 + 1$  formulation established in *Chapter 2*, this appendix shall provide a comprehensive annotation of the concept.

**Definition 20** *Let  $M$  be a smooth manifold. A **congruence** is a family of non-intersection curves<sup>1</sup> which fills a region  $V \subseteq M$ , i.e. each point  $p \in V$  lies on one and only one curve. The congruence is generated by a vector field which is tangential to each point on a curve of the congruence. They are said to be the integral curves of the generator.*

Let  $\lambda \in \mathbb{R}$  be the *parameter* of one curve. Consider a point on the curve at the parameter value  $p(\lambda_0)$ . Then, a congruence defines a map such that  $p(\lambda_0)$  can be mapped into a point  $p(\lambda_0 + \Delta\lambda)$  a bit “further away” but still on this curve. Hence, a congruence defines a one-to-one map in a region  $V$ , where the *generating vector field* is well-defined and at least  $\mathcal{C}^1$ . If the generator is even  $\mathcal{C}^\infty$  then the map is a diffeomorphism. The figurative explanation already suggests that this map provides the *dragging of points* along the congruence. Therefore this map is commonly called *Lie dragging* [36].

The dragging permits the definition of a *derivative along the congruence*, although the concept of a derivation of vectors and tensors on a general manifold is a difficulty in itself [42]. In Euclidean geometry a natural *parallelism* exists and therefore vectors at two different points can easily be compared with each other. In a curved space there is no natural identification of the tangent spaces at two distinguishable points. In order to compare vectors or tensors living in different spaces and calculate their infinitesimal difference as we do in simple analysis, an additional structure has to be added:

---

<sup>1</sup>A smooth curve  $C$  on the manifold  $M$  is simply a  $\mathcal{C}^\infty$ -map from  $I \subseteq \mathbb{R}$  into  $M$ .



an *affine connection* (see [42, 36]). But, using the frame work of a congruence, the problem of parallelism can be avoided in an elegant way: vectors and/or tensors defined at different points along the congruence can easily be dragged and evaluated at the *same* point. If a vector defined at the point  $p(\lambda_0)$  is Lie dragged along the congruence to the point  $p(\lambda_0 + \Delta\lambda)$  and evaluated there, a new vector arises. The comparison between this new one and the original one defined at this point yields a unique difference and hence defines the infinitesimal change of the vector which is the fundamental idea of a derivative. But notice that this difference does, of course, depend on the choice of the congruence. This kind of derivative along a congruence is called the **Lie derivative** and commonly denoted by the special operator  $\mathcal{L}$ .

Let us now derive an explicit expression for this kind of derivative. First consider a scalar function  $f$  defined on  $V \subseteq M$ . The procedure suggested above is the following: evaluate the function  $f$  at the parameter value  $\lambda_0 + \Delta\lambda$ , drag it back to the point at  $\lambda_0$ , subtract the value of the function there, divide it by the parameter difference  $\Delta\lambda$  and take the limit  $\Delta\lambda \rightarrow 0$  [36]. Notice that the back dragging or *pullback* defines a new function  $\tilde{f}$  at  $\lambda_0$ . Hence, the result is:

$$\left. \frac{df}{d\lambda} \right|_{\lambda_0} = \lim_{\Delta\lambda \rightarrow 0} \frac{\tilde{f}(\lambda_0) - f(\lambda_0)}{\Delta\lambda} \quad (\text{A.1})$$

Thinking about calculus and the difference quotient, the above result is, of course, not surprising. As expected, the Lie derivative of a function can be generalised to the Lie derivative of vectors, one-forms and eventually to any kind of tensors.

In order to see the link between the general definition of the Lie derivative and our application to the 3 + 1 formalism immediately, we change our notation: let  $\varphi_t$  be the congruence generated by the vector field  $\xi \in \mathcal{X}(V)$ , where  $t \in \mathbb{R}$  is the curve parameter and  $\mathcal{X}(V)$  denotes the set of the smooth vector fields on  $V \subseteq M$ . Due to Definition 20 it is clear, that  $\xi$  is defined as the derivative of the congruence, hence satisfying a set of ordinary differential equations, namely  $\xi := \dot{\varphi}$ . Further, let  $\mathbf{T}$  be a tensor field of type  $(k, l)$  defined on this region  $V$  in  $M$  and  $p$  the point on the curve with parameter value  $t = 0$ . Then, the **Lie derivative** of  $\mathbf{T}$  along  $\xi$  is given by

$$(\mathcal{L}_\xi \mathbf{T})(p) = \lim_{\Delta t \rightarrow 0} \left\{ \frac{(\varphi_{-\Delta t}^* \mathbf{T})(p) - \mathbf{T}(p)}{\Delta t} \right\} \quad (\text{A.2})$$

or equivalently by

$$\mathcal{L}_\xi \mathbf{T} := \left. \frac{d}{dt} \right|_{t=0} (\varphi_t^* \mathbf{T}), \quad (\text{A.3})$$

where the star-operation denotes the dragging or pullback as introduced in *Chapter 2*. This means that the Lie derivative describes the negative rate of change of a smooth tensor field when being Lie dragged along a vector field. Further it follows immediately from the above definition that the Lie derivative  $\mathcal{L}_\xi$  is a linear map from smooth tensor fields of type  $(k, l)$  to smooth tensor fields of type  $(k, l)$ , it permutes with the contraction and satisfies the Leibniz rule [42]. Hence, it is a derivation on the tensor algebra.

In order to analyse the action of the Lie derivative on an arbitrary tensor field, it is helpful to introduce local coordinates  $\{x^\mu\}$  on  $M$  such that the family parameter  $t$  is chosen to be one of the coordinates, e.g.  $x^0$ . The generating vector field  $\xi$  can then be chosen to be the tangent to the coordinate line  $\frac{\partial}{\partial x^0}$ . This choice of coordinates is called an *adapted coordinate system*. The action  $\varphi_{-\Delta t}^*$  then corresponds to a simply coordinate transformation, namely to  $x^0 \mapsto x^0 + \Delta t$  [42]. Consequently, the components of the Lie derivative of  $\mathbf{T}$  written in a coordinate system adapted to the generating vector field  $\xi$  are:

$$\mathcal{L}_\xi \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x^0}. \quad (\text{A.4})$$

In adapted coordinates the Lie derivative of a vector field  $\mathbf{u}$  respectively its components is then given by

$$\mathcal{L}_\xi u^\sigma = \frac{\partial u^\sigma}{\partial x^0}. \quad (\text{A.5})$$

From this we can obtain a coordinate independent expression for the Lie derivative of a vector field  $\mathbf{u}$  by considering that  $u = u^\alpha \frac{\partial}{\partial x^\alpha}$ . Hence, it follows

$$\mathcal{L}_\xi \mathbf{u} = [\xi, \mathbf{u}], \quad (\text{A.6})$$

where the bracket denotes the common commutator of the two vector fields. Notice, that Eq.(A.6) is a coordinate-independent formula and hence valid in any arbitrary coordinate system and thus holds for any derivative operator, e.g. the partial derivate, the covariant derivative etc. [42]. The same procedure applies to the derivation of the Lie derivative of the dual vector field. Thus, one finds the final expression for the Lie derivative of an arbitrary tensor field:

$$\begin{aligned} \mathcal{L}_\xi T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = & \xi^\sigma \nabla_\sigma T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} - \sum_{i=1}^k T^{\mu_1 \dots \sigma \dots \mu_k}_{\nu_1 \dots \nu_l} \nabla_\sigma \xi^{\mu_i} \\ & + \sum_{j=1}^l T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \sigma \dots \nu_l} \nabla_{\nu_j} \xi^\sigma \end{aligned} \quad (\text{A.7})$$

*Remark 1:* It is rather important to notice, that the Lie derivative is only

a *local concept*. The integral curves of the generating vector field only have to exist up to a finite value of the family parameter which is equivalent to  $\xi$  being only well-defined in a certain region of the manifold  $M$ .

*Remark 2:* Regarding the explicit form of the generating vector field (see Eq.(2.65)) certain generic points may occur. One might wonder, if the Lie derivative is even defined in such cases. The answer is that one does not have to worry about nulls or purely spatial  $\xi$  if and only if the considered tensor field is tangential to the initial hypersurface  $\varphi_0(\Sigma)$ .

## Appendix B

# Time derivative of the unit normal $\mathbf{n}$

This chapter is dedicated to the proofs of the essential relations in *Chapter 2* concerning the time derivative of the unit normal vector  $\mathbf{n}$ . Therefore, let us start the following:

Consider two vectors  $X, Y \in \mathcal{T}_p\varphi_t(\Sigma)$ . Due to the considerations of *Chapter 2* we know that

$$X^\mu = \varphi_{,j}^\mu c^j, \quad Y^\mu = \varphi_{,j}^\mu d^j \quad (\text{B.1})$$

and further that

$$(X^\lambda \nabla_\lambda Y^\mu)_{||} = \varphi_{,j}^\mu c^i D_i d^j, \quad (\text{B.2})$$

where “ $||$ ” denotes the part parallel to the hypersurface  $\varphi_t(\Sigma)$ . Consequently, the normal part is defined by

$$(X^\lambda \nabla_\lambda Y^\mu)_\perp = -n^\mu n_\nu X^\lambda \nabla_\lambda Y^\nu. \quad (\text{B.3})$$

Furthermore, it is well-known that the complete derivative of  $Y$  along  $X$  is given by

$$X^\lambda \nabla_\lambda Y^\mu = (X^\lambda \nabla_\lambda Y^\mu)_{||} + (X^\lambda \nabla_\lambda Y^\mu)_\perp. \quad (\text{B.4})$$

In order to derive an explicit expression of the above equation in terms of  $\varphi_{,k}^\sigma$  we first have to rewrite Eq.(B.3). This yields

$$\begin{aligned} -n^\mu n_\nu X^\lambda \nabla_\lambda Y^\nu &= X^\lambda \nabla_\lambda (n^\mu n_\nu) Y^\nu \\ &= \varphi_{,j}^\nu d^j \varphi_{,i}^\lambda c^i \nabla_\lambda (n^\mu n_\nu) \\ &= -K_{ij} n^\mu c^i d^j. \end{aligned} \quad (\text{B.5})$$

Plugging this in Eq.(B.4) as well as the definition of the covariant derivative it follows

$$\begin{aligned}
\varphi_{,j}^\mu c^i D_i d^j - n^\mu K_{ij} c^i d^j &= X^\lambda \partial_\lambda Y^\mu + \Gamma^\mu_{\alpha\lambda} Y^\alpha \\
&= \varphi_{,i}^\lambda c^i \partial_\lambda (\varphi_{,j}^\mu d^j) + \Gamma^\mu_{\alpha\lambda} \varphi_{,i}^\lambda \varphi_{,j}^\alpha c^i d^j \\
&= c^i \partial_k (\varphi_{,j}^\mu d^j) + \Gamma^\mu_{\alpha\lambda} \varphi_{,i}^\lambda \varphi_{,j}^\alpha c^i d^j \\
&= \varphi_{,ij}^\mu c^i d^j + c^i \varphi_{\mu j} \partial_i d^j + \Gamma^\mu_{\alpha\lambda} \varphi_{,i}^\lambda \varphi_{,j}^\alpha c^i d^j.
\end{aligned} \tag{B.6}$$

Rewriting the induced covariant derivative  $D_i$  on the left-hand side as

$$D_i d^j = \partial_i d^j + \bar{\Gamma}_{ki}^j d^k \tag{B.7}$$

then yields immediately the desired result:

$$n^\mu K_{ij} = -\varphi_{,ij}^\mu - \Gamma^\mu_{\alpha\lambda} \varphi_{,i}^\lambda \varphi_{,j}^\alpha + \bar{\Gamma}_{ij}^k \varphi_{,k}^\mu \tag{B.8}$$

since all the ingredients are valid  $\forall c, d \in \mathcal{T}_p(\Sigma)$ .

Now, differentiating the orthonormality as well as the normalisation condition w.r.t. the time parameter  $t$  gives

$$g_{\mu\nu, \lambda} \dot{\varphi}^\lambda n^\mu n^\nu + 2g_{\mu\nu} \dot{n}^\mu n^\nu = 0 \tag{B.9}$$

and

$$\dot{\varphi}_{,i}^\mu n_\mu + \varphi_{,i}^\mu \dot{n}_\mu = 0, \tag{B.10}$$

where the “dot” symbolises  $\partial_t$  and the comma denotes the partial coordinate derivative. Before we can start with the actual derivation of  $\dot{n}$  we have to express  $\dot{\varphi}_{,i}^\mu \equiv \xi_{,i}^\mu$ :

$$\dot{\varphi}_{,i}^\mu = (N n^\mu + \varphi_{,j}^\mu \beta^j)_{,i} = N_{,i} n^\mu + N n_{,i}^\mu + \varphi_{,ij}^\mu \beta^j + \varphi_{,j}^\mu \beta_{,i}^j. \tag{B.11}$$

Contracting the above equation with  $n_\mu$  then gives

$$n_\mu \dot{\varphi}_{,i}^\mu = -N_{,i} + N n_\mu n_{,i}^\mu + n_\mu \varphi_{,ij}^\mu \beta^j. \tag{B.12}$$

Plugging Eq.(B.8) into Eq.(B.12) and inserting this expression into Eq.(B.10) then yields

$$-N_{,i} + N n_\mu n_{,i}^\mu + K_{ij} \beta^j - n_\mu \Gamma^\mu_{\alpha\lambda} \varphi_{,i}^\lambda \varphi_{,j}^\alpha \beta^j + \varphi_{,i}^\mu \dot{n}_\mu = 0, \tag{B.13}$$

where we have again used the orthonormality condition. In order to determine  $\dot{n}$  we make the following general *Ansatz*:

$$\dot{n}^\mu = \alpha n^\mu + \varphi_{,k}^\mu X^k, \tag{B.14}$$

with  $\alpha \in \mathbb{R}$  and  $X \in \mathcal{T}_p(\Sigma)$ . Contraction with  $g_{\mu\nu}\varphi_{,k}^\nu$  then gives

$$\dot{n}^\mu g_{\mu\nu}\varphi_{,k}^\nu = h_{jk}X^k \equiv X_j. \quad (\text{B.15})$$

and contraction with  $n_\mu$  yields an expression for the proportionality factor  $\alpha$ , namely:

$$\alpha = \frac{1}{2}g_{\mu\nu,\lambda}\dot{\varphi}^\lambda n^\mu n^\nu, \quad (\text{B.16})$$

where we made use of Eq.(B.9). Now, we are well-equipped to derive an explicit expression for the time derivative of the unit normal  $\mathbf{n}$ . Starting with Eq.(B.13) we find:

$$\begin{aligned} 0 &= -N_{,i} + Nn_\mu n_{,i}^\mu + K_{ij}\beta^j - n_\mu \Gamma_{\alpha\lambda}^\mu \varphi_{,i}^\lambda \varphi_{,j}^\alpha \beta^j + \varphi_{,i}^\mu \partial_t(g_{\mu\nu}n^\nu) \\ &= -N_{,i} + Nn_\mu n_{,i}^\mu + K_{ij}\beta^j - n_\mu \Gamma_{\alpha\lambda}^\mu \varphi_{,i}^\lambda \varphi_{,j}^\alpha \beta^j + \\ &\quad + \varphi_{,j}^\mu g_{\mu\nu,\lambda}\dot{\varphi}^\lambda n^\nu + \varphi_{,i}^\mu g_{\mu\nu}\dot{n}^\nu. \end{aligned} \quad (\text{B.17})$$

The last term is given by Eq.(B.15) and the term  $Nn_\mu n_{,i}^\mu$  turns out to be the following when starting with the normalisation condition:

$$\begin{aligned} \partial_i(g_{\mu\nu}n^\mu n^\nu) &= g_{\mu\nu,\lambda}\varphi_{,i}^\lambda n^\mu n^\nu + 2g_{\mu\nu}n_{,i}^\mu n^\nu = 0 \\ \Rightarrow g_{\mu\nu,\lambda}\varphi_{,i}^\lambda n^\mu n^\nu &= -2n_\mu n_{,i}^\mu \\ \Rightarrow Nn_\mu n_{,i}^\mu &= -\frac{1}{2}N g_{\mu\nu,\lambda}\varphi_{,i}^\lambda n^\mu n^\nu. \end{aligned} \quad (\text{B.18})$$

Hence, all together with have by now

$$X_i = N_{,i} + \frac{1}{2}g_{\mu\nu,\lambda}\varphi_{,i}^\lambda n^\mu n^\nu - K_{ij}\beta^j + n_\mu \Gamma_{\alpha\lambda}^\mu \varphi_{,i}^\lambda \varphi_{,j}^\alpha \beta^j - \varphi_{,i}^\mu g_{\mu\nu,\lambda}\dot{\varphi}^\lambda n^\nu. \quad (\text{B.19})$$

Inserting this into our *Ansatz* Eq.(B.14) after transforming  $X_i$  into  $X^i$ , we get:

$$\begin{aligned} \dot{n}^\mu &= \frac{1}{2}g_{\rho\nu,\lambda}\dot{\varphi}^\lambda n^\rho n^\nu + \varphi_{,i}^\mu D^i N - h^{\mu\rho}g_{\rho\nu,\lambda}\dot{\varphi}^\lambda n^\nu - \\ &\quad - \varphi_{,i}^\mu K_{,j}^i \beta^j + h^{\mu\alpha}\Gamma_{\alpha\lambda}^\rho \varphi_{,j}^\lambda \beta^j n_\rho + \frac{1}{2}N g_{\rho\nu,\lambda}h^{\mu\lambda}n^\rho n^\nu. \end{aligned} \quad (\text{B.20})$$

If one now uses the explicit form of the Christoffel symbols

$\Gamma_{\alpha\lambda}^\rho = \frac{1}{2}g^{\rho\sigma}(g_{\sigma\alpha,\lambda} + g_{\sigma\lambda,\alpha} - g_{\alpha\lambda,\sigma})$  we find the final expression for the time derivative of the unit normal:

$$\dot{n}^\mu + \Gamma_{\alpha\lambda}^\mu n^\alpha \dot{\varphi}^\lambda = \varphi_{,i}^\mu (D^i N - K_{,j}^i \beta^j). \quad (\text{B.21})$$

Let us further take a look at the following object in order to interpret the time evolution of  $\mathbf{n}$  in a general slicing:

$$\begin{aligned} \xi^\lambda \nabla_\lambda n^\mu &= Nn^\lambda \partial_\lambda n^\mu + Nn^\lambda \Gamma_{\alpha\lambda}^\mu n^\alpha + \beta^\lambda \partial_\lambda n^\mu + \beta^\lambda \Gamma_{\alpha\lambda}^\mu n^\alpha \\ &= Nn^\lambda \partial_\lambda n^\mu + \beta^\lambda \partial_\lambda + \Gamma_{\alpha\lambda}^\mu \dot{\varphi}^\lambda n^\alpha \\ &= \dot{\varphi}^\lambda \partial_\lambda n^\mu + \Gamma_{\alpha\lambda}^\mu \dot{\varphi}^\lambda n^\alpha. \end{aligned} \quad (\text{B.22})$$

Hence, we conclude that *if* the unit normal vector field  $\mathbf{n}$  is defined on  $M$ , i.e. if  $n = n(\varphi(t, y^i))$  respectively  $N > 0$ , then the above relation directly corresponds to the time derivative of  $\mathbf{n}$  by the chain rule.

## Appendix C

# Evolution equation for $\mathbf{K}$ - an alternative derivation

This section is dedicated to an alternative but mathematically interesting derivation of the evolution equation of the extrinsic curvature  $\mathbf{K}$ . It appears like a little trick to multiply Einstein's equations with the lapse function  $N$  and try to extract the time derivative of  $\mathbf{K}$  via constructing a Lie derivative somehow “artificially” as we have already done in *Chapter 4*, but this necessary step was well motivated in *Chapter 2*. Here, we deduce the evolution equation for  $\mathbf{K}$  via the mathematically more reasonable way, namely by

$$\partial_t K_{ij} = \varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\xi K_{\mu\nu}. \quad (\text{C.1})$$

Though one does not see directly the connection to the field equations, from the initial value formulation we already know that  $\mathbf{K}$  is the second dynamical variable besides the spatial metric. Unfortunately, this way is more challenging as it requires a good knowledge of the Lie derivative and commutator relations between the covariant and the Lie derivative. Luckily, the comprehensive book [44] by Kentaro Yano contains all the relevant relations.<sup>1</sup> Furthermore, we assume again  $N > 0$  in this derivation. Hence, we are perfectly equipped to perform this general derivation of the evolution equation.

Let us start with an essential commutator relation, known as the *Lie bracket*, which can be found in [44] p.16:

$$\mathcal{L}_\xi(\nabla_\mu n_\nu) - \nabla_\mu \mathcal{L}_\xi n_\nu = -(\mathcal{L}_\xi \Gamma_{\mu\nu}^\rho) n_\rho. \quad (\text{C.2})$$

Since Christoffel symbols associated with the spacetime metric  $\mathbf{g}$  appear in the above relation, we further have to find an explicit expression for the

---

<sup>1</sup>Yano uses a different sign convention than we do. In order to use the equations one has to check for consistency.



right-hand side. Fortunately, this important relation can also be found in Yano's book on p.8:

$$\mathcal{L}_\xi \Gamma^\rho_{\mu\nu} = \nabla_\mu \nabla_\nu \xi^\rho + {}^4R_{\mu\sigma\nu}{}^\rho \xi^\sigma \quad (\text{C.3})$$

Further we need to compute the second term on the left-hand side of Eq.(C.2):

$$\begin{aligned} \nabla_\mu \mathcal{L}_\xi n_\nu &= \nabla_\mu \{ \xi^\lambda \nabla_\lambda n_\nu + N_\lambda \nabla_\nu \xi^\lambda \} \\ &= n_\nu \nabla_\mu (n^\lambda \nabla_\lambda N + \beta^\lambda a_\lambda) + (n^\lambda \nabla_\lambda N + \beta^\lambda a_\lambda) \nabla_\mu n_\nu \end{aligned} \quad (\text{C.4})$$

Now, the projection of the above result as needed for Eq.(C.1) yields

$$\varphi^\mu_{,i} \varphi^\nu_{,j} \nabla_\mu \mathcal{L}_\xi n_\nu = n^\rho (\mathcal{L}_\xi n_\rho) K_{ij}. \quad (\text{C.5})$$

Taking into account what we have by now, we get the following important intermediate result:

$$\begin{aligned} \dot{K}_{ij} &= -\varphi^\mu_{,i} \varphi^\nu_{,j} (-n_\rho \mathcal{L}_\xi \Gamma^\rho_{\mu\nu} + \nabla_\mu \mathcal{L}_\xi n_\nu) \\ &= \varphi^\mu_{,i} \varphi^\nu_{,j} n_\rho \mathcal{L}_\xi \Gamma^\rho_{\mu\nu} - K_{ij} n^\rho \mathcal{L}_\xi n_\rho \\ &= \varphi^\mu_{,i} \varphi^\nu_{,j} n_\rho \{ \nabla_\mu \nabla_\nu \xi^\rho + {}^4R_{\mu\sigma\nu}{}^\rho \xi^\sigma \} - K_{ij} n^\rho \mathcal{L}_\xi n_\rho. \end{aligned} \quad (\text{C.6})$$

Making use of  $\mathcal{L}_\xi n_\nu = (n^\lambda \nabla_\lambda N + \beta^\lambda a_\lambda) n_\nu$  we get

$$\begin{aligned} \dot{K}_{ij} &= \varphi^\mu_{,i} \varphi^\nu_{,j} n_\lambda \{ \nabla_\mu \nabla_\nu (N n^\lambda + \beta^\lambda) + {}^4R_{\mu\sigma\nu}{}^\lambda \xi^\sigma \} + \\ &\quad + K_{ij} n^\lambda \nabla_\lambda N + K_{ij} a_k \beta^k \end{aligned} \quad (\text{C.7})$$

There are still some steps before we reach our final result. Therefore, we compute the complex expressions in Eq.(C.7) separately. An important relation we often make use of in the following is:

$$\nabla_\lambda N = D_\lambda N - (n^\rho \nabla_\rho N) n_\lambda \quad (\text{C.8})$$

Notice that the above equation is the natural connection between the covariant derivative associated with the spacetime metric  $\mathbf{g}$  and the one associated with  $\mathbf{h}$ . It is a direct consequence of the unique decomposition of the tangent space of  $M$ .

We can now focus on the explicit computation of the remaining terms. We start with the first term on the right-hand side of Eq.(C.7):

$$\begin{aligned} \varphi^\mu_{,i} \varphi^\nu_{,j} n_\lambda \nabla_\mu \{ n^\lambda \nabla_\nu N + N \nabla_\nu n^\lambda + \nabla_\nu \beta^\lambda \} &= \\ &= \varphi^\mu_{,i} \varphi^\nu_{,j} n_\lambda \{ n^\lambda \nabla_\mu \nabla_\nu N + N \nabla_\mu \nabla_\nu n^\lambda + \nabla_\mu \nabla_\nu \beta^\lambda \} \\ &= \varphi^\mu_{,i} \varphi^\nu_{,j} \{ -\nabla_\mu \nabla_\nu N - n_\lambda N \nabla_\mu K_\nu{}^\lambda + n_\lambda \nabla_\mu \nabla_\nu \beta^\lambda \} \\ &= \varphi^\mu_{,i} \varphi^\nu_{,j} \{ -\nabla_\mu D_\nu N + \nabla_\mu [(n^\lambda \nabla_\lambda N) n_\nu] + N K_\nu{}^\lambda \nabla_\mu n_\lambda + n_\lambda \nabla_\mu \nabla_\nu \beta^\lambda \} \\ &= -D_i D_j N - K_{ij} n^\lambda \nabla_\lambda N - N K_{im} K_j{}^m + \varphi^\mu_{,i} \varphi^\nu_{,j} n_\lambda \nabla_\mu \nabla_\nu \beta^\lambda, \end{aligned} \quad (\text{C.9})$$

where we have used  $K_\mu^\lambda n_\lambda = 0$  as well as Eq.(C.8) and the orthonormality condition Eq.(2.17). Therefore, we get the following mean result:

$$\begin{aligned}\dot{K}_{ij} = & -D_i D_j N - N K_{im} K_j^m + \varphi_{,i}^\mu \varphi_{,j}^\nu n_\lambda \nabla_\mu \nabla_\nu \beta^\lambda + K_{ij} a_k \beta^k + \\ & + \varphi_{,i}^\mu \varphi_{,j}^\nu N g_{\lambda\rho} {}^4 R_{\mu\sigma\nu}^\lambda n^\rho n^\sigma + \varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,k}^\sigma n_\lambda {}^4 R_{\mu\sigma\nu}^\lambda \beta^k\end{aligned}\quad (\text{C.10})$$

Unfortunately, the above equation still contains three terms which have to be handled further. The next term we work out is the last one on the right side of Eq.(C.10):

$$\begin{aligned}\varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,k}^\sigma g_{\rho\lambda} n^\rho {}^4 R_{\mu\sigma\nu}^\lambda \beta^k = \\ = \varphi_{,i}^\mu \varphi_{,j}^\nu \varphi_{,k}^\sigma n^\rho {}^4 R_{\mu\sigma\nu\rho} \beta^k \\ = \beta^k D_k K_{ij} - \beta^k D_i K_{jk}.\end{aligned}\quad (\text{C.11})$$

In order to retrieve the final line of Eq.(C.11) we have used the Codazzi equation Eq.(2.44) as well as the pair symmetry of the Riemann curvature tensor.

Let us treat the second term containing the Riemann tensor in Eq.(C.10) next:

$$\begin{aligned}\varphi_{,i}^\mu \varphi_{,j}^\nu N n^\rho n^\sigma {}^4 R_{\mu\sigma\nu\rho} = \\ = \varphi_{,i}^\mu \varphi_{,j}^\nu N (h^{\rho\sigma} - g^{\rho\sigma}) {}^4 R_{\mu\sigma\nu\rho} \\ = \varphi_{,i}^\mu \varphi_{,j}^\nu N h^{\rho\sigma} {}^4 R_{\rho\nu\sigma\mu} - \varphi_{,i}^\mu \varphi_{,j}^\nu N {}^4 R^\sigma_{\nu\sigma\mu} \\ = N^3 R_{ij} + \mathcal{K} K_{ij} - K_{im} K_j^m - N \varphi_{,i}^\mu \varphi_{,j}^\nu {}^4 R_{\mu\nu},\end{aligned}\quad (\text{C.12})$$

where we made use of Eq.(2.33), the definition of the Ricci tensor and the contracted Gauß relation Eq.(2.54).

Based on  $\varphi_{,i}^\mu \varphi_{,j}^\nu \nabla_\mu \nabla_\nu (n_\lambda \beta^\lambda) = 0$  we are ready to work out the very last term:

$$\begin{aligned}\varphi_{,i}^\mu \varphi_{,j}^\nu n_\lambda \nabla_\mu \nabla_\nu \beta^\lambda = \\ = -\varphi_{,i}^\mu \varphi_{,j}^\nu \{ (\nabla_\mu \beta^\lambda) \nabla_\nu n_\lambda + \beta^\lambda \nabla_\mu \nabla_\nu n_\lambda + (\nabla_\mu n_\lambda) \nabla_\nu \beta^\lambda \} \\ = \varphi_{,i}^\mu \varphi_{,j}^\nu \{ K_{\nu\lambda} \nabla_\mu \beta^\lambda + K_{\mu\lambda} \nabla_\nu \beta^\lambda + \beta^\lambda \nabla_\mu K_{\nu\lambda} + \beta^\lambda a_\lambda \nabla_\mu n_\nu \} \\ = K_{jk} D_i \beta^k + K_{ik} D_j \beta^k + \beta^k D_i K_{kj} - \beta^k a_k K_{ij}\end{aligned}\quad (\text{C.13})$$

Finally, we can collect all the results from the auxiliary calculations Eq.(C.11)-Eq.(C.13) and plug them into the original equation. Notice that the terms containing the scalar product of the 4-acceleration and the shift vector annihilate whereas the terms  $K_{im} K_j^m$  add up. Further, we know already that the projection of the Ricci tensor is equal to the projection of the right-hand side of the Einstein equations Eq.(5.3) yielding Eq.(5.9). Thus, we are able

to write down the ultimate result as already known from the less elegant but maybe more obvious derivation in *Chapter 5*:

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)K_{ij} = & -D_i D_j N - 2N K_{im} K_j^m + N \mathcal{K} K_{ij} + \\
& + N^3 R_{ij} - 4\pi N (S_{ij} - {}^4\mathcal{T} h_{ij}).
\end{aligned} \tag{C.14}$$

*Remark:* In order to deduce this essential equation in the setting of a completely general slicing, we would not be allowed to insert  $K_{\mu\nu} = -\nabla_\mu n_\nu$  into Eq.(C.1) as we do here, since the identification of the time derivative with the pullback of the Lie derivative along  $\xi$  can no longer be made for non-tangent objects. Hence, we would have to start the following way:

$$\begin{aligned}
\dot{K}_{ij} &= \varphi_{,i}^\mu \varphi_{,j}^\nu \mathcal{L}_\xi K_{\mu\nu} \\
&= \varphi_{,i}^\mu \varphi_{,j}^\nu (\xi^\lambda \nabla_\lambda K_{\mu\nu} + K_{\mu\lambda} \nabla_\nu \xi^\lambda + K_{\lambda\nu} \nabla_\mu \xi^\lambda),
\end{aligned} \tag{C.15}$$

respectively by

$$\dot{K}_{ij} = \partial_t (\varphi_{,i}^\mu \varphi_{,j}^\nu K_{\mu\nu}). \tag{C.16}$$

## Appendix D

# The stress-energy tensor

This additional chapter is dedicated to the right-hand side of Einstein's field equations Eq.(5.1) - the *stress-energy tensor*  $\mathbf{T}$ .

As already known from Special Relativity (see e.g. [22]), continuous mass distributions are described by the symmetric second rank tensor, the *stress-energy tensor*. It is the generalisation of the mass density and therefore, in the context of General Relativity, the source of the gravitational field. In order to interpret the components of the stress-energy tensor, we first have to introduce a reference frame respectively an observer. We choose an observer with 4-velocity  $\mathbf{u}$ . By definition, its length is  $g(u, u) = -1$  with respect to a Lorentzian metric  $\mathbf{g}$  of signature  $(-, +, +, +)$  as considered here. For an observer with 4-velocity  $\mathbf{u}$  the  $T^{00}$  component is interpreted as the *energy density* as measured by this observer:

$$\epsilon := T(u, u). \quad (\text{D.1})$$

Chosen a coordinate system  $\{x^\sigma\}$  on  $M$ , this component is given by  $\epsilon = T_{\mu\nu}u^\mu u^\nu$ . For baryonic matter this quantity  $\epsilon$  is always nonnegative, hence  $T_{\mu\nu}u^\mu u^\nu \geq 0$  [42].

Furthermore, it follows directly from the very definition of the stress-energy tensor, that the  $T^{0i}$  components describe the *momentum density* of the matter:

$$p := -T(u, w), \quad (\text{D.2})$$

where the vector  $\mathbf{w}$  is orthogonal<sup>1</sup> to the 4-velocity, hence  $g(u, w) = 0$ . The components  $-T_{\mu\nu}u^\mu w^\nu$  are interpreted as the momentum density as measured by the observer characterised by the 4-velocity  $\mathbf{u}$ .

---

<sup>1</sup>In the context of the initial value formulation  $\mathbf{w}$  simply represents a vector tangent to the considered hypersurface or the projection of a general vector onto  $\Sigma$ .

Since  $\mathbf{T}$  is a symmetric tensor of rank 2, the missing components are the spatial-spatial ones  $T_{ij}$ . This sub-tensor is commonly denoted by a different letter, namely by  $\mathbf{S}$  and is called the *stress-tensor*. In particular the  $S_{ii}$  components represent the normal stress or pressure as measured by this observer. We conclude that in general the quantities of the stress-energy tensor as measured by an observer with 4-velocity  $\mathbf{u}$  can be written as

$$T_{\mu\nu} = \begin{pmatrix} \epsilon & -p_1 & -p_2 & -p_3 \\ -p_1 & p & S_{12} & S_{13} \\ -p_2 & S_{21} & p & S_{23} \\ -p_3 & S_{31} & S_{32} & p \end{pmatrix}.$$

Obviously, the stress-energy tensor reduces to the rather simple diagonal matrix  $\text{diag}(\epsilon, p, p, p)$  in the momentarily comoving reference frame of the matter. Taking the trace of the energy-stress tensor with respect to the spacetime metric  $\mathbf{g}$ , as required in Eq.(5.8), yields

$$g^{\mu\nu}T_{\mu\nu} = \mathcal{T} = S - \epsilon, \quad (\text{D.3})$$

where we have used  $u^\mu u_\mu = -1$  as well as  $p_\mu u^\mu = 0$ . “S” denotes the trace of the stress tensor.

For our purposes we also introduce the physical quantities defined above as measured by an observer with 4-velocity  $\mathbf{n}$ . Thus, the below-mentioned relations follow immediately:

$$\epsilon = T_{\mu\nu} n^\mu n^\nu \quad (\text{D.4})$$

$$p_i = -n^\mu \varphi_{,i}^\nu T_{\mu\nu} \quad (\text{D.5})$$

$$S_{ij} = \varphi_{,i}^\mu \varphi_{,j}^\nu T_{\mu\nu}. \quad (\text{D.6})$$

Hence, the standard 3 + 1 decomposition of the stress-energy tensor writes as

$$T_{\mu\nu} = S_{\mu\nu} + n_\mu p_\nu + p_\mu n_\nu + \epsilon n_\mu n_\nu. \quad (\text{D.7})$$



# Bibliography

- [1] Miguel Alcubierre: *Introduction to 3+1 Numerical Relativity*, Oxford University Press 2008
- [2] Miguel Alcubierre, Juan Carlos Degollado & Marcelo Salgado: *The Einstein-Maxwell system in 3+1 form and initial data for multiple charged black holes*, Phys. Rev. D 80, 104022 (2009), arXiv:0907.1151v2 [gr-qc]
- [3] A.M. Anile: *Relativistic fluids and magneto-fluids with applications in astrophysics and plasma physics*, Cambridge University Press, 1989
- [4] Robert Beig & Mark J. Heinzle: *CMC-Slicings of Kottler-Schwarzschild-de Sitter Cosmologies*, Commun. Math. Phys. **260**, 673-709, 2005
- [5] Robert Beig & Neill O’Murchadha: *Late time behaviour of the maximal slicing of the Schwarzschild black hole*, Phys.Rev. D **57**, 4728-4737, 1998
- [6] Robert Beig & Piotr T. Chrusciel: *Killing Initial Data*, Class.Quant.Grav.**14**, A83-A92, 1997
- [7] R.A. Bertlmann: *Anomalies in Quantum Field Theory*, Oxford University Press, 2000
- [8] J. David Brown: *Action functionals for relativistic perfect fluids*, Class. Quantum Grav. **10** (1993) 1579
- [9] Yvonne Choquet-Bruhat: *General Relativity and the Einstein Equations*, Oxford University Press 2009
- [10] Demetrios Christodoulou: *Self-Gravitating Relativistic Fluids: A Two-Phase Model*, Arch. Rational Mech. Anal. **130** (1995) 343-400
- [11] R. Courant & D. Hilbert: *Methoden der mathematischen Physik*, Springer, 4. Auflage, 1993
- [12] P.A. Davidson: *An Introduction to Magnetohydrodynamics*, Cambridge University Press, 2001

- [13] Ruth Durrer & Norbert Straumann: *Some applications of the 3+1 formalism of General Relativity*, Helv. Phys. Acta **61**, 1027, 1988
- [14] R.A.Bauer & J- Ehlers: *Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space-time. I*, Proc. R. Soc. Lond. A **370**, 389-406 (1980)
- [15] Lawrence C. Evans: *Partial Differential Equations*, American Math. Soc., 1998
- [16] José A. Font: *Numerical Hydrodynamics and Magnetohydrodynamics in General Relativity*, Living Rev. Relativity 11, (2008), 7. URL (cited on 29.4.2010): <http://www.livingreviews.org/lrr-2008-7>
- [17] Yvonne Fourès-Bruhat (Choquet-Bruhat): *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Mathematica **88**, 141 (1952)
- [18] Yvonne Fourès-Bruhat (Choquet-Bruhat): *Sur l'intégration des équations de la relativité générale*, J. Rational Mech. Anal. **5**, 951 (1956)
- [19] Jörg Frauendiener: *A note on the relativistic Euler equations*, Class. Quantum Grav. **20** (2003) L193-L196
- [20] Éricourgoulhon: *3+1 Formalism and Bases of Numerical Relativity*, arXiv:gr-gc/0703035v1, 6.3.2007
- [21] Donald Greenspan: *Introduction to Partial Differential Equations*, Dover Publications Inc., Mineola, New York, 1961
- [22] Walter Greiner: *Theoretische Physik Band 3A: Spezielle Relativitätstheorie*, Verlag Harri Deutsch, 3. Auflage, 1992
- [23] Walter Greiner: *Klassische Elektrodynamik*, Verlag Harri Deutsch, 6. Auflage, 2002
- [24] S.W.Hawking & G.F.R. Ellis: *The large scale structure of space-time*, Cambridge University Press 1973
- [25] J. Mark Heinzle: *Relativity and Cosmology I*, Lecture Notes, version 20.1.2010, <http://gravity.univie.ac.at>
- [26] A.V. Kats & J. Juul Rasmussen: *Variational principle for the relativistic hydrodynamic flows with discontinuities, and local invariants of motion*, arXiv:physics/0407019v1 [physics.flu-dyn]
- [27] Lev D. Landau & Evgenij M. Lifšic: *Hydrodynamik, Lehrbuch der Theoretischen Physik Band 6*, Akademischer Verlag Berlin, 2. Auflage, 1971



- [28] André Lichnerowicz: *Relativistic Hydrodynamics and Magnetohydrodynamics*, Benjam New York, 1967
- [29] André Lichnerowicz: *Sur certains problèmes globaux relatifs au système des équations d'Einstein*, Hermann, Paris (1939), Actual. Sci. Ind. **833**
- [30] André Lichnerowicz: *L'intégration des équations de la gravitation relativiste et le problème des  $n$  corps*, J. Math. Pures Appl. **23**, 31 (1944)
- [31] André Lichnerowicz: *Sur les équations relativistes de la gravitation*, Bulletin de la S.M.F. **80**, 237 (1952)
- [32] C.W. Misner, K.S. Thorne & J.A. Wheeler: *Gravitation*, W.H. Freeman and Company, New York, 1973
- [33] Barrett O'Neill: *Semi-Riemannian Geometry with Applications to Relativity*, Acad. Pr. 1993, 4<sup>th</sup> print run
- [34] Reinhard Prix: *Variational description of multi-fluid hydrodynamics: Uncharged fluids*, Phys.Rev. D **69** (2004) 043001, arXiv:physics/0209024v3 [physics.flu-dyn]
- [35] Alan D. Rendall: *Theorems on Existence and Global Dynamics for the Einstein Equations*, Living Rev. Relativity, **8**, (2005), 6. [Online Article]: cited [4.7.2010], <http://www.livingreviews.org/lrr-2005-6>
- [36] Bernard Schutz: *Geometrical Methods of Mathematical Physics*, Cambridge University Press, 1980
- [37] Norbert Straumann: *General Relativity with Applications to Astrophysics*, Springer Verlag Berlin, 2004
- [38] A.H. Taub: *On Circulation in Relativistic Hydrodynamics*, Arch. Rational Mech. Anal. **3** (1959) 312-324
- [39] Kip S. Thorne & Douglas MacDonald: *Electrodynamics in curved space-time: 3+1 formulation*, MNRAS (1982) **198**, 339-343
- [40] Kip S. Thorne et. al.: *Cauchy problem in spacetimes with closed timelike curves*, Phys. Rev. D **42**, 19151930 (1990)
- [41] Roman Sexl & Helmuth K. Urbantke: *Relativity, Groups, Particles*, Springer, 2001
- [42] Robert M. Wald: *General Relativity*, The University of Chicago Press 1984
- [43] Steven Weinberg: *Gravitation and Cosmology*, John Wiley and Sons, Inc. 1972

- [44] Kentaro Yano: *The Theory of Lie Derivatives and its Applications*, North-Holland Publishing Co., Amsterdam 1955

# Curriculum Vitae

## Persönliche Daten

Name	Patricia Schmidt
Geburtstag	7. Februar 1986
Geburtsort	Wien

## Ausbildung

1992-1996	Besuch der Volksschule Dr. Albert-Geßmann-Gasse in Wien
1996-2004	Besuch des BRG Polgarstraße in Wien mit Schwerpunkt Mathematik und Naturwissenschaften
06/2004	Matura mit ausgezeichnetem Erfolg
10/2004	Beginn des Diplomstudiums Physik an der Universität Wien
10/2004	Beginn des Bakkalaureatsstudiums Astronomie an der Universität Wien
01/2006	Abschluss der ersten Diplomprüfung Physik mit ausgezeichnetem Erfolg
01/2009	Abschluss der zweiten Diplomprüfung Physik mit ausgezeichnetem Erfolg
06/2009	Abschluss des Bakkalaureatsstudiums Astronomie mit ausgezeichnetem Erfolg durch die Bakkalaureatsarbeit "Probing General Relativity with Gravitational Waves" unter der Betreuung von Prof. Dr. Dieter Breitschwerdt
07/2009	Beginn der Diplomarbeit bei Prof. Dr. Beig in der Fachgruppe Gravitationsphysik
01/2010	Abschluss des ersten Teils der dritten Diplomprüfung Physik mit ausgezeichnetem Erfolg

## Weitere berufliche und wissenschaftliche Erfahrungen

07/2007-08/2007	Forschungspraktikum am Max Planck Institut für Extraterrestrische Physik, Garching/München
-----------------	--

10/2007-03/2008	Tutorin des Frauentutoriums Physik an der Universität Wien
06/2008	Stipendiengeförderte Teilnahme am LISA Symposium, IEEC, Barcelona
07/2008-09/2008	Forschungspraktikum am Max Planck Institut für Gravitationsphysik, Hannover
03/2009-07/2009	Tutorin zur Vorlesung “Scientific Computing” an der Universität Wien
09/2009	Teilnahme am Intensivkurs “On the black hole information paradox” von Dr. Samir Mathur, SISSA, Triest
03/2010-07/2010	Tutorin zur Vorlesung “Mathematische Methoden der Physik I”